Products of Distributions in Several Variables and Applications to Zero-Mass QED_2

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Abstract. We study products of distributions in serveral variables, having in mind applications to quantum electrodynamics. We introduce a new product, the parameter product, and relate it to known ones. It allows us to rigorously interpret and evaluate products arising in the computations of the one-loop vacuum polarization of zero-mass QED_2, thereby avoiding the occurrence of renormalization ambiguities from the very beginning.

Key words: *Products of distributions, renormalization, Wightman distributions*

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1. Introduction

In this article we study products of distributions in several variables, having in mind certain applications to quantum electrodynamics in two-dimensional Minkowski space-time (QED_2). We mainly focus on distributions which are smooth functions of one variable with values in the space of distributions of the other variables, the two-point Wightman distributions being of this type. We introduce and study a new product involving such distributions, the parameter product, establish its relationship with known distributional products, and use it to rigorously evaluate certain products arising in QED_2.

We now give some background on how these problems arise in quantum electrodynamics: The main goal of renormalized perturbative quantization is to give meaning to the time ordered products of free fields' Wick monomials as they emerge from the formal expression for the scattering operator [10, 18]. In the physical literature, the problem is posed and solved through extending the time ordered products, initially defined on the open set of non-coinciding points, to all values of the space-time variables. The ambiguity inherent in this procedure is then eliminated through imposing renormalization conditions. It has been known since long [3] that interacting quantum fields in two-dimensional (2D) space-time permit a formulation in terms of an interaction picture, which is due to the less singular character of the Wick monomials of free 2D quantum fields. This fact has made possible a computation of the time ordered vacuum expectation value $TJ^{\mu\nu}(x;0,2)$ of the two-point Wightman distribution $J^{\mu\nu}(x;0,2)$ of the vector current of a free, zero-mass Dirac quantum field according to the naive formula [6, 12]:

$$TJ^{\mu\nu}(x;m^2=0,n=2)=\Theta(x^0)J^{\mu\nu}(x;0,2)+\Theta(-x^0)J^{\nu\mu}(-x;0,2)$$

instead of following the usually employed renormalization procedure. Here the distribution $J^{\mu\nu}$ has to be multiplied with the Heaviside function Θ, while in the definition of $J^{\mu\nu}$ itself products of distributions are encountered, which do not have an a priori well-defined meaning. Up to now the notion of these products has not been made sufficiently precise, thus leaving gaps in the existing attempts to eliminate the renormalization ambiguities from $TJ^{\mu\nu}$. The main application in our paper is that these computations, as presented in [6, 12], are brought into a rigorous form by specifying the occurring distributional products in an unambiguous way and computing them accordingly.

The paper is organized as follows: Section 2 serves to fix notation and to provide the necessary background on multiplication of distributions. We specially collect the relevant facts about multiplication via Fourier transform and convolution, termed the Fourier product here. In Section 3, the parameter product is introduced: it applies to smoothly parametrized distributions, to be multiplied with a distribution in the parameter variable. We show that if the parameter product exists, so does the Fourier product, and the results coincide. However, the parameter product, if applicable, is much more effective from a computational viewpoint. In particular, the Leibniz rule for differentiation holds. This section also contains a study of products of two-dimensional distributions which are translates of one-dimensional ones. In Section 4 we present the applications to quantum electrodynamics. We first derive the explicit coordinate space formula for the Wightman two-point distribution of the vector current of the zero-mass Dirac quantum field by means of an interplay between the parameter and the Fourier product. Then we compute the time ordered two-point distribution $TJ^{\mu\nu}$, in which example the computational advantages of the parameter product become apparent.

Apart from the usual notation from distribution theory, we conform with the notational conventions of relativistic quantum field theory. In particular, the variables in coordinate space (n-dimensional Minkowski space) are denoted by $x = (x^0, \boldsymbol{x}) = (x^0, x^1, ..., x^{n-1})$, in momentum space by $k = (k^0, \boldsymbol{k})$. For the placement of the Fourier product in a hierarchy of more general distributional products we have to refer to the literature [13, 15, 16]. We wish to thank P. Wagner for valuable discussions.

2. Basic facts about multiplication of distributions

One of the common methods for multiplying distributions, frequently encountered in the physical literature, consists in going over to momentum space via Fourier transform and performing a convolution. If the convolution integral converges, its inverse Fourier transform gives the value of the product. The class of distributions which can be multiplied this way is greatly enlarged if the procedure above is preceded by a localization in coordinate space. This kind of product has first been rigorously established by Ambrose [1], reformulated in [15]. Before introducing it in detail, let us recall some facts about convolution.

For $n \geq 1$ we denote by $\mathscr{D}'(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$, by $\mathscr{S}'(\mathbb{R}^n)$ the space of tempered distributions, and by $\mathscr{D}'_{L^1}(\mathbb{R}^n)$ the space of integrable distributions. The latter consists of those distributions which can be written as a finite sum of derivatives of integrable functions. The integrable distributions are those which can be applied to any smooth function on $\mathbb{R}^n$ which is bounded together with all its derivatives. The Fourier transform of tempered distributions will be defined by means of the Minkowski metric $k \cdot x := k^0 x^0 - \boldsymbol{k} \cdot \boldsymbol{x}$ for $k = (k^0, \boldsymbol{k})$ and $x = (x^0, \boldsymbol{x}) \in \mathbb{R}^n$:

$$\langle \mathscr{F} U, \varphi \rangle := \langle U, \mathscr{F} \varphi \rangle, \quad \mathscr{F} \varphi(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} d^n x \; \varphi(x) \; e^{-ik \cdot x}$$

for $U \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then the exchange formula is given by $U \circ V = (2\pi)^{-n/2} \mathscr{F}^{-1}(\mathscr{F} U * \mathscr{F} V)$ if the products and convolutions do make sense.

Given $U, V \in \mathscr{S}'(\mathbb{R}^n)$, the $\mathscr{S}'$-convolution of U and V is said to exist iff for all test functions $\varphi \in \mathscr{S}(\mathbb{R}^n)$

$$(\varphi * \check{U}) V \in \mathscr{D}'_{L^1}(\mathbb{R}^n), \tag{2.1}$$

where $\check{U}$ is defined by $\check{U}(x) := U(-x)$ and $(\varphi * \check{U})(x) := \langle \check{U}(y), \varphi(x - y) \rangle$. Equivalently, one could require (cf. Theorem (2.3) of Ref. [5]) that

$$\varphi(x + y) U(x) \otimes V(y) \in \mathscr{D}'_{L^1}(\mathbb{R}^{2n}) \tag{2.2}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The $\mathscr{S}'$-convolution of U and V is then defined by

$$\big\langle U * V, \varphi \big\rangle := \big\langle (\varphi * \check{U}) V, 1 \big\rangle = \big\langle \varphi(x + y) U(x) \otimes V(y), 1 \big\rangle, \tag{2.3}$$

where 1 denotes the constant function equal to one. Conditions (2.1) and (2.2) guarantee that $U*V$ is a continuous functional on $\mathscr{S}(\mathbb{R}^n)$ and thus belongs to $\mathscr{S}'(\mathbb{R}^n)$ [11], which is not necessarily true if other definitions are used; see the example in [5]. Definition (2.3) generalizes both the convolution of integrable functions as well as the convolution of two tempered distributions whose supports are in favorable position: More precisely, let H be a closed half-space in $\mathbb{R}^n$ with interior normal N and let Γ be a closed, convex, acute cone in $\mathbb{R}^n$ with axial vector pointing in the direction of N (in particular, the intersection of Γ with the complement of H is relatively compact). Denote by $\mathscr{S}'(\Gamma)$ the set of tempered distributions having their support in Γ. The following basic result can be proved by the arguments of [22, Chap. I, § 5.6]:

Proposition 2.1: *Let H and Γ be as above. The $\mathscr{S}'$-convolution has the following properties:*

(a) If $U \in \mathscr{S}'(H)$ and $V \in \mathscr{S}'(\Gamma)$, then the $\mathscr{S}'$-convolution of U and V exists.

(b) If both U and V belong to $\mathscr{S}'(\Gamma)$, then $U * V \in \mathscr{S}'(\Gamma)$ as well.

(c) The convolution map $* : \mathscr{S}'(\Gamma) \times \mathscr{S}'(\Gamma) \to \mathscr{S}'(\Gamma)$ is continuous with respect to the topologies induced by the strong topology of $\mathscr{S}'(\mathbb{R}^n)$.

We are now ready to present the precise definition of multiplication of distributions by localization and Fourier transform [1, 15]. Let U, V belong to $\mathscr{D}'(\mathbb{R}^n)$. Assume that for every $x \in \mathbb{R}^n$ there is a neighborhood Ω_x and $f_x \in \mathscr{D}(\mathbb{R}^n)$ with $f_x \equiv 1$ on Ω_x so that the $\mathscr{S}'$-convolution of $\mathscr{F}(f_x U)$ and $\mathscr{F}(f_x V)$ exists. In this case we say that the *Fourier product* of U and V exists in $\mathscr{D}'(\mathbb{R}^n)$; it is defined by

$$\left\langle U \underset{\mathrm{F}}{\circ} V, \varphi \right\rangle := (2\pi)^{-n/2} \sum_{j=1}^{\infty} \left\langle \mathscr{F}^{-1}\left[\mathscr{F}(f_{x_j} U) * \mathscr{F}(f_{x_j} V)\right], \chi_j \varphi \right\rangle \tag{2.4}$$

for $\varphi \in \mathscr{D}(\mathbb{R}^n)$, where $\{\chi_j \mid j \in \mathbb{N}\}$ is a locally finite smooth partition of unity subordinate to the cover $\{\Omega_x \mid x \in \mathbb{R}^n\}$ with $\operatorname{supp}\chi_j \subset \Omega_{x_j}$. It can be shown that this definition does not depend on the choice of the Ω_{x_j}, f_{x_j} and χ_j.

Important special cases arise when U and V have disjoint singular supports, when the wave front sets of U and V are in favorable position [17], or when U and V are tempered and the supports of their Fourier transforms are in favorable position, cf. [15]. In the latter case we say that the Fourier product exists in the *strong sense* and formula (2.4) simplifies to

$$U \underset{\mathrm{F}}{\circ} V = (2\pi)^{-n/2} \mathscr{F}^{-1}(\mathscr{F}U * \mathscr{F}V). \tag{2.5}$$

Note that $U \underset{\mathrm{F}}{\circ} V$ then necessarily belongs to $\mathscr{S}'(\mathbb{R}^n)$.

As an example for the advantage of the preceding localization, consider the product of two Dirac measures. Let $n = 1$ and $a, b \in \mathbb{R}$, $a \neq b$. Then the attempt to define $\delta(x-a)\delta(x-b)$ by (2.5) leads to the divergent integral $\int_{-\infty}^{\infty} dk\, e^{i(a-b)k}$, whereas formula (2.4) immediately yields $\delta(x-a) \underset{\mathrm{F}}{\circ} \delta(x-b) = 0$. The product does not exist when $a = b$.

For reference in Section 4, let us recall the following well-known example in dimension $n = 1$. Let

$$\delta_{\pm}(x) := (x \pm i\epsilon)^{-1}\big|_{\epsilon\downarrow 0} = \mathrm{Pf}\,\frac{1}{x} \mp i\pi\delta(x).$$

Here $\mathrm{Pf}\,\frac{1}{x} := \frac{d}{dx}\ln|x|$ denotes the "partie finie" of $\frac{1}{x}$. The Fourier transform of $\delta_+(x)$ is $-2\pi i\Theta(k)$; its support is obviosly $\mathbb{R}_0^+ := [0, \infty)$. Therefore, the square of $\delta_+(x)$ exists as Fourier product according to formula (2.5), and we have

$$\delta_+(x) \underset{\mathrm{F}}{\circ} \delta_+(x) = \mathrm{Pf}\,\frac{1}{x^2} + i\pi\delta'(x) = (x + i\epsilon)^{-2}\big|_{\epsilon\downarrow 0}. \tag{2.6}$$

Similarly,

$$\delta_-(x) \underset{\mathrm{F}}{\circ} \delta_-(x) = \mathrm{Pf}\,\frac{1}{x^2} - i\pi\delta'(x) = (x - i\epsilon)^{-2}\big|_{\epsilon\downarrow 0}\,. \tag{2.7}$$

Using the consistency result of Proposition 2.2 below, these fomulae may also be established by analytic continuation along the lines of [8, Chap. I, Sec. 3.6, and Chap. II, Sec. 2.3].

We note that if the Fourier product of two distributions U and V exists, it can also be computed by means of regularization and passage to the limit. To be precise, let us call a net $\{\varphi_\epsilon\}_{\epsilon>0}$ of test functions $\varphi_\epsilon \in \mathscr{D}(\mathbb{R}^n)$ a *strict delta-net*, provided

supp $\varphi_\epsilon \to \{0\}$ as $\epsilon \to 0$, $\int \varphi_\epsilon(x)\; d^n x = 1$, $\forall \epsilon > 0$,

and $\int |\varphi_\epsilon(x)|\; d^n x$ is bounded independently of ϵ.

Such a net converges to the Dirac measure $\delta^n(x)$.

Proposition 2.2: *Let $U, V \in \mathscr{D}'(\mathbb{R}^n)$. If the Fourier product of U and V exists and $\{\varphi_\epsilon\}_{\epsilon>0}$, $\{\psi_\epsilon\}_{\epsilon>0}$ are strict delta-nets, then all the following limits exist in $\mathscr{D}'(\mathbb{R}^n)$ and coincide with the product:*

$$U \underset{\mathrm{F}}{\circ} V = \lim_{\epsilon\to 0}\, (U * \varphi_\epsilon)V = \lim_{\epsilon\to 0} U(V * \psi_\epsilon) = \lim_{\epsilon\to 0}\, (U * \varphi_\epsilon)(V * \psi_\epsilon).$$

Proofs of Proposition 2.2 have been given in [4, 15, 21]; in the case where the product exists in the strong sense, already in [11]. We remark that the existence of the limits, even for all strict delta-nets, does not imply the existence of the Fourier product in general. Regularization and passage to the limit actually furnishes a more general way of defining a product of distributions.

The Fourier product has the following useful properties: Assume that the Fourier product of two distributions U and V exists and that f is a smooth function. Then the Fourier products of (fU) and V and of U and (fV) exist as well, and

$$(fU) \underset{\mathrm{F}}{\circ} V = U \underset{\mathrm{F}}{\circ} (fV) = f\;(U \underset{\mathrm{F}}{\circ} V) \tag{2.8}$$

holds [19]. Also, let us denote by $\tau_h : \mathbb{R}^n \to \mathbb{R}^n$ the translation by $h \in \mathbb{R}^n$. Defining $\tau_h U$ by $\langle \tau_h U, \varphi\rangle := \langle U, \tau_{-h}\varphi\rangle$ for $\varphi \in \mathscr{D}(\mathbb{R}^n)$, where $(\tau_{-h}\varphi)(x) := \varphi(x+h)$, one checks without difficulty that

$$(\tau_h U) \underset{\mathrm{F}}{\circ} (\tau_h V) = \tau_h(U \underset{\mathrm{F}}{\circ} V). \tag{2.9}$$

3. The parameter product

This section is devoted to distributions which may be viewed as smooth functions of one variable with values in the space of distributions of the other variables. It will be convenient to denote the "parameter-variable" by t. The set of maps $t \mapsto U(t;x)$ where $U(t;x)$ belongs to $\mathscr{D}'(\mathbb{R}^n)$ for fixed t and the dependence on t is smooth (i.e. $t \mapsto \langle U(t;x), \varphi(x)\rangle$ is smooth for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$) will be denoted by $\mathscr{C}^\infty(\mathbb{R} : \mathscr{D}'(\mathbb{R}^n))$.

Let $U \in \mathscr{C}^\infty(\mathbb{R} : \mathscr{D}'(\mathbb{R}^n))$ and $V \in \mathscr{D}'(\mathbb{R})$. Then the *parameter product* of U and V is defined as a distribution belonging to $\mathscr{D}'(\mathbb{R}^{n+1})$ as follows:

$$\Big\langle V \underset{\mathrm{p}}{\circ} U, \varphi\Big\rangle := \Big\langle V(t), \big\langle U(t;\cdot), \varphi(t,\cdot)\big\rangle\Big\rangle \tag{3.1}$$

for $\varphi \in \mathscr{D}(\mathbb{R}^{n+1})$. In order to show that this definition is meaningful, one verifies without difficulty that the map $t \mapsto \langle U(t;\cdot), \varphi(t,\cdot)\rangle$ is a smooth function with compact support. One has the formula

$$\frac{d}{dt}\big\langle U(t;\cdot), \varphi(t,\cdot)\big\rangle = \left\langle \frac{d}{dt}U(t;\cdot), \varphi(t,\cdot)\right\rangle + \left\langle U(t;\cdot), \frac{\partial}{\partial t}\varphi(t,\cdot)\right\rangle. \tag{3.2}$$

An application of (3.2) shows that the Leibniz rule holds for the parameter product,

$$\frac{\partial}{\partial t}(V \underset{\mathrm{P}}{\circ} U) = \Big(\frac{d}{dt}V\Big) \underset{\mathrm{P}}{\circ} U + V \underset{\mathrm{P}}{\circ} \Big(\frac{d}{dt}U\Big). \tag{3.3}$$

Also,

$$\frac{\partial}{\partial x^j}(V \underset{\mathrm{P}}{\circ} U) = V \underset{\mathrm{P}}{\circ} \Big(\frac{\partial}{\partial x^j}U\Big) \quad \text{for } j = 1, \ldots, n. \tag{3.4}$$

The following identification will be needed. To any continuous function $U(t;x)$ of $t \in \mathbb{R}$ with values in $\mathscr{D}'(\mathbb{R}^n)$ one can associate a distribution $\iota U \in \mathscr{D}'(\mathbb{R}^{n+1})$ by means of the prescription

$$\langle \iota U, \varphi\rangle := \int_{-\infty}^{\infty} \langle U(t;\cdot), \varphi(t,\cdot)\rangle \, dt$$

for $\varphi \in \mathscr{D}(\mathbb{R}^{n+1})$. This assignment $\iota : \mathscr{C}(\mathbb{R} : \mathscr{D}'(\mathbb{R}^n)) \hookrightarrow \mathscr{D}'(\mathbb{R}^{n+1})$ is an imbedding of the continuous $\mathscr{D}'(\mathbb{R}^n)$-valued functions into $\mathscr{D}'(\mathbb{R}^{n+1})$. If no confusion can arise, we subsequently shall not distinguish between U and ιU in our notation.

Proposition 3.1: *Let $U \in \mathscr{C}^\infty(\mathbb{R} : \mathscr{D}'(\mathbb{R}^n))$, $V \in \mathscr{D}'(\mathbb{R})$. Then the Fourier product of $(\iota U)(t,x)$ and $V(t) \otimes 1(x)$ exists, and $(V \otimes 1) \underset{\mathrm{F}}{\circ} \iota U = V \underset{\mathrm{P}}{\circ} U$.*

Proof: In order to avoid notational complications due to the Minkowski metric, we give the proof only for the case $n = 1$; the same arguments apply in the general case. It will be convenient to denote the variables in $\mathbb{R}^2$ by (t,x), on the Fourier transform side by (τ,ξ). We are going to show that the $\mathscr{S}'$-convolution of $\mathscr{F}_{(t,x)}[(\psi \otimes \psi)(V \otimes 1)]$ and $\mathscr{F}_{(t,x)}[(\psi \otimes \psi)\iota U]$ exists for every $\psi \in \mathscr{D}(\mathbb{R})$.

To estimate the first term, we write $[\psi(t) \otimes \psi(x)][V(t) \otimes 1(x)] = W(t) \otimes \psi(x)$ with $W(t) := \psi(t)V(t) \in \mathscr{E}'(\mathbb{R})$. Since W has compact support, its Fourier transform grows at most polynomially, while $\mathscr{F}_x\psi$ is rapidly decreasing. Thus there are some $k \in \mathbb{N}$ and constants $c_m > 0$ such that

$$\big|\mathscr{F}_{(t,x)}(W \otimes \psi)(\tau,\xi)\big| = \big|\mathscr{F}_t W(\tau) \otimes \mathscr{F}_x\psi(\xi)\big| \le c_m(1+|\tau|)^k(1+|\xi|)^{-m}$$

for every $m \in \mathbb{N}$. By Peetre's inequality [20, p. 17], an additional convolution with a rapidly decreasing function does not change these growth properties. Thus, for all $\varphi \in \mathscr{S}(\mathbb{R}^2)$,

$$\big|\varphi * \mathscr{F}_{(t,x)}(W \otimes \psi)^\vee(\tau,\xi)\big| \le c_m(1+|\tau|)^k(1+|\xi|)^{-m}$$

for every $m \in \mathbb{N}$, where we possibly had to increase the constants c_m, depending on φ.

Next, we estimate the second term $[\psi(t) \otimes \psi(x)]\iota U(t,x) =: G(t,x)$. Clearly, G is a smooth and compactly supported map of the variable t with values in $\mathscr{E}'(\mathbb{R})$, i.e. $G \in \mathscr{D}(\mathbb{R} : \mathscr{E}'(\mathbb{R}))$. In particular,

$$\big\{\frac{d^j}{dt^j}G(t,\cdot) \mid 0 \le j \le k+2, t \in \mathbb{R}\big\}$$

is a bounded, even a compact subset of $\mathscr{E}'(\mathbb{R})$, and hence uniformly bounded on some neighborhood of zero in $\mathscr{E}(\mathbb{R})$, say on

$$\mathscr{N} = \big\{\chi \in \mathscr{E}(\mathbb{R}) \mid \sup_{j \le l}\sup_{x \in K}\big|\frac{d^j}{dt^j}\chi(x)\big| \le 1\big\}$$

where $l \in \mathbb{N}$ and $K \subset \mathbb{R}$ is compact. We observe that the functions $x \mapsto \epsilon(1+|\xi|)^{-l}e^{-ix\xi}$ belong to $\mathscr{N}$ for sufficiently small ϵ and all $\xi \in \mathbb{R}$, so

$$\big|\frac{d^j}{dt^j}\mathscr{F}_x G(t,\xi)\big| = (2\pi)^{-\frac{1}{2}}\Big|\big\langle\frac{d^j}{dt^j}G(t,x), e^{-ix\xi}\big\rangle\Big| \le C\,\epsilon^{-1}(1+|\xi|)^l$$

for some $C > 0$, all $t \in \mathbb{R}$ and $0 \le j \le k+2$. We note also that $t \mapsto \mathscr{F}_x G(t,\cdot)$ is a smooth, compactly supported function with values in $\mathscr{E}(\mathbb{R})$. Finally, it follows that

$$\left|\tau^j \big(\mathscr{F}_{(t,x)} G\big)(\tau,\xi)\right| = (2\pi)^{-\frac{1}{2}} \left| \int_{-\infty}^{\infty} e^{-it\tau} \frac{d^j}{dt^j} \mathscr{F}_x G(t,-\xi)\ dt \right| \le C\ \epsilon^{-1}(1+|\xi|)^l$$

for some $C > 0$ and $0 \le j \le k+2$. We conclude that

$$\left|\mathscr{F}_{(t,x)}\big[(\psi\otimes\psi)\iota U\big](\tau,\xi)\right| \le C(1+|\xi|)^l (1+|\tau|)^{-k-2}.$$

Putting the estimates together, we obtain

$$\begin{aligned} &\left|\Big(\varphi * \mathscr{F}_{(t,x)}\big[(\psi\otimes\psi)(V\otimes 1)\big]^{\vee} \mathscr{F}_{(t,x)}\big[(\psi\otimes\psi)\iota U\big]\Big)(\tau,\xi)\right| \\ &\qquad \le c_m C(1+|\xi|)^{-m}(1+|\tau|)^k(1+|\xi|)^l(1+|\tau|)^{-k-2} \end{aligned}$$

for all $m \in \mathbb{N}$. Therefore, this product belongs to $L^1(\mathbb{R}^2) \subset \mathscr{D}'_{L^1}(\mathbb{R}^2)$, so the $\mathscr{S}'$-convolution of $\mathscr{F}_{(t,x)}[(\psi\otimes\psi)(V\otimes 1)]$ and $\mathscr{F}_{(t,x)}[(\psi\otimes\psi)\iota U]$ exists, and so does the Fourier product $(V\otimes 1) \underset{\mathrm{F}}{\circ} \iota U$.

To prove the equality with the parameter product $V \underset{\mathrm{p}}{\circ} U$, we use Proposition 2.2. Let $\{\varphi_\epsilon\}_{\epsilon>0}$ be a strict delta-net in $\mathscr{D}(\mathbb{R}^{n+1})$. Then $\psi_\epsilon(t) := \int_{\mathbb{R}^n} \varphi_\epsilon(t,x)\ d^n x$ is a one-dimensional strict delta-net, and $(V\otimes 1) * \varphi_\epsilon = (V * \psi_\epsilon)\otimes 1$. Thus, for any $\varphi \in \mathscr{D}(\mathbb{R}^{n+1})$,

$$\begin{aligned} \Big\langle \big[(V\otimes 1)*\varphi_\epsilon\big]\iota U, \varphi\Big\rangle &= \int_{-\infty}^{\infty} \Big\langle U(t;\cdot), \big[(V\otimes 1)*\varphi_\epsilon\big](t,\cdot)\ \varphi(t,\cdot)\Big\rangle\ dt \\ &= \int_{-\infty}^{\infty} \big(V*\psi_\epsilon\big)(t)\ \big\langle U(t;\cdot), \varphi(t,\cdot)\big\rangle\ dt \\ &= \Big\langle \big(V*\psi_\epsilon\big)(t), \big\langle U(t;\cdot),\varphi(t,\cdot)\big\rangle\Big\rangle. \end{aligned}$$

But $V*\psi_\epsilon \to V$ in $\mathscr{D}'(\mathbb{R})$ as $\epsilon \to 0$ and, as indicated above, $\langle U(t;\cdot),\varphi(t,\cdot)\rangle \in \mathscr{D}(\mathbb{R})$. Thus the last term converges to $V \underset{\mathrm{p}}{\circ} U$, while the first term converges to $(V\otimes 1)\underset{\mathrm{F}}{\circ} \iota U$. ∎

Proposition 3.1 shows that if the parameter product is applicable, it gives the same result as the Fourier product and thus may be used to facilitate the calculations. We note that, in general, the wave front sets of ιU and $V\otimes 1$ need not be in favorable position. For example, let $U(t,x) := \sum_{k=1}^{\infty} 2^{-k}\delta(x-kt)$. Then $U \in \mathscr{C}^\infty(\mathbb{R} : \mathscr{D}'(\mathbb{R}))$, but nevertheless $(0,0,-1,0)$ belongs to the wave front set of ιU. On the other hand, $(0,0,1,0)$ belongs to the wave front set of $\delta(t)\otimes 1(x)$, so the wave front set criterion is not applicable, while the parameter and Fourier products of ιU and $\delta\otimes 1$ actually do exist.

A second important case of products involving distributions depending on a parameter arises in dimension $n = 2$: it concerns two-dimensional distributions which are translates of one-dimensional ones. We are going to show that their two-dimensional product may be calculated as a one-dimensional product with the other variable fixed in certain circumstances.

Proposition 3.2: *Let $f, g \in \mathscr{S}'(\mathbb{R})$ and assume that their Fourier transforms vanish on a common half-axis; let $\alpha, \beta \in \mathbb{R}$. Then the distributions $U(t;x) := f(x+\alpha t)$, $V(t;x) := g(x+\beta t)$ belong to $\mathscr{S}'(\mathbb{R}^2)$, and their two-dimensional Fourier transforms have supports in a common acute cone. The Fourier product of U and V, viewed as two-dimensional distributions, exists in the strong sense. Further, the one-dimensional Fourier product $U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot)$ exists for every fixed t and defines a smooth map $\mathbb{R} \to \mathscr{S}'(\mathbb{R})$. Finally, the products coincide:*

$$U \underset{\mathrm{F}_{(t,x)}}{\circ} V = \iota\big[U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot)\big] \in \mathscr{S}'(\mathbb{R}^2).$$

Proof: Again, we denote the variables on $\mathbb{R}^2$ by (t,x), on the Fourier transform side by (τ,ξ). We first study the one-dimensional Fourier product of $U(t;\cdot)$ and $V(t;\cdot)$ for fixed t. Observe that $\mathscr{F}_x U(t;\xi) = e^{i\alpha t\xi}(\mathscr{F}_x f)(\xi)$ and $\mathscr{F}_x V(t;\xi) = e^{i\beta t\xi}(\mathscr{F}_x g)(\xi)$ hold. Let χ be a smooth function, identically one on the half-axis supporting $\mathscr{F}_x f$ and $\mathscr{F}_x g$, and zero off some distance to this half-axis. Let $\varphi \in \mathscr{S}(\mathbb{R})$. Clearly, the $\mathscr{S}'$-convolution of $\mathscr{F}_x U(t;\cdot)$ and $\mathscr{F}_x V(t;\cdot)$ exists, and

$$\begin{aligned}
&\left\langle U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot), \varphi \right\rangle \\
&= (2\pi)^{-\frac{1}{2}} \left\langle \mathscr{F}_x U(t;\cdot) * \mathscr{F}_x V(t;\cdot), \mathscr{F}_x^{-1}\varphi \right\rangle \\
&= (2\pi)^{-\frac{1}{2}} \left\langle \mathscr{F}_x^{-1}\varphi(\xi+\eta)\big[e^{i\alpha t\xi}\mathscr{F}_x f(\xi) \otimes e^{i\beta t\eta}\mathscr{F}_x g(\eta)\big], 1 \right\rangle \\
&= (2\pi)^{-\frac{1}{2}} \left\langle \mathscr{F}_x f(\xi) \otimes \mathscr{F}_x g(\eta), e^{it(\alpha\xi+\beta\eta)}\chi(\xi)\chi(\eta)\mathscr{F}_x^{-1}\varphi(\xi+\eta) \right\rangle .
\end{aligned}$$

But the map $t \mapsto e^{it(\alpha\xi+\beta\eta)}\chi(\xi)\chi(\eta)\mathscr{F}_x^{-1}\varphi(\xi+\eta)$ is smooth with values in $\mathscr{S}(\mathbb{R}^2)$. Thus the last equality shows that $\langle U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot), \varphi\rangle$ depends smoothly on t, i.e. $U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot) \in \mathscr{C}^\infty(\mathbb{R} : \mathscr{S}'(\mathbb{R}))$. The second to the last equality shows that

$$\begin{aligned}
&\left\langle \iota\big[U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot)\big], \psi \right\rangle \\
&= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left\langle \mathscr{F}_x^{-1}\psi(t,\xi+\eta)\big[e^{it\alpha\xi}\mathscr{F}_x f(\xi) \otimes e^{it\beta\eta}\mathscr{F}_x g(\eta)\big], 1 \right\rangle dt
\end{aligned}$$

for $\psi \in \mathscr{D}(\mathbb{R}^2)$.

Now we turn to the two-dimensional Fourier product of U and V. A straightforward calculation shows that, for $\psi \in \mathscr{S}(\mathbb{R}^2)$,

$$\begin{aligned}
\left\langle \mathscr{F}_{(t,x)} U, \psi \right\rangle &= (2\pi)^{\frac{1}{2}} \left\langle \mathscr{F}_x f(\xi), \psi(\alpha\xi, -\xi) \right\rangle, \\
\left\langle \mathscr{F}_{(t,x)} V, \psi \right\rangle &= (2\pi)^{\frac{1}{2}} \left\langle \mathscr{F}_x g(\xi), \psi(\beta\xi, -\xi) \right\rangle.
\end{aligned}$$

In particular, the supports of $\mathscr{F}_{(t,x)}U$ and $\mathscr{F}_{(t,x)}V$ are contained in the half-rays $\{(\tau,\xi) \in \mathbb{R}^2 \mid -\xi \in \operatorname{supp}\mathscr{F}_x f, \tau = -\alpha\xi\}$ and $\{(\tau,\xi) \in \mathbb{R}^2 \mid -\xi \in \operatorname{supp}\mathscr{F}_x g, \tau = -\beta\xi\}$, respectively. It is clear that we can find an acute cone Γ containing both half-rays. Thus the two-dimensional Fourier product of U and V exists in the strong sense. Letting χ be a smooth cut-off function identically one on Γ and taking $\psi \in \mathscr{S}(\mathbb{R}^2)$, we have

$$\begin{aligned}
&\left\langle U \underset{\mathrm{F}_{(t,x)}}{\circ} V, \psi \right\rangle \\
&= (2\pi)^{-1} \left\langle \mathscr{F}_{(t,x)}U * \mathscr{F}_{(t,x)}V, \mathscr{F}_{(t,x)}^{-1}\psi \right\rangle \\
&= (2\pi)^{-2} \Big\langle \mathscr{F}_{(t,x)}U(\tau,\xi) \otimes \mathscr{F}_{(t,x)}V(\sigma,\eta), \\
&\qquad\qquad \chi(\tau,\xi)\chi(\sigma,\eta) \int_{\mathbb{R}^2} e^{it(\tau+\sigma)-ix(\xi+\eta)}\psi(t,x)\,dx\,dt \Big\rangle \\
&= (2\pi)^{-1} \Big\langle \mathscr{F}_x f(\xi) \otimes \mathscr{F}_x g(\eta), \\
&\qquad\qquad \chi(\alpha\xi,-\xi)\chi(\beta\eta,-\eta) \int_{\mathbb{R}^2} e^{it(\alpha\xi+\beta\eta)+ix(\xi+\eta)}\psi(t,x)\,dx\,dt \Big\rangle \\
&= (2\pi)^{-\frac{1}{2}} \Big\langle \mathscr{F}_x f(\xi) \otimes \mathscr{F}_x g(\eta) \otimes 1(t), \\
&\qquad\qquad \chi(\alpha\xi,-\xi)\chi(\beta\eta,-\eta)e^{it(\alpha\xi+\beta\eta)}\mathscr{F}_x^{-1}\psi(t,\xi+\eta) \Big\rangle \\
&= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left\langle \mathscr{F}_x^{-1}\psi(t,\xi+\eta)\left[e^{it\alpha\xi}\mathscr{F}_x f(\xi) \otimes e^{it\beta\eta}\mathscr{F}_x g(\eta)\right], 1 \right\rangle dt.
\end{aligned}$$

The integrand in the last equality is obtained by observing that the cut-offs $\chi(\alpha\xi, -\xi)$ and $\chi(\beta\eta, -\eta)$ are identically equal to 1 on the half-axis supporting $\mathscr{F}_x f$ and $\mathscr{F}_x g$. At last, it shows that $U \underset{\mathrm{F}_{(t,x)}}{\circ} V = \iota\big[U(t;\cdot) \underset{\mathrm{F}_x}{\circ} V(t;\cdot)\big]$. ∎

4. Applications to QED$_2$

In this section we shall apply our results to rigorously calculating certain products arising in quantum electrodynamics. We start by considering a free, mass-m Dirac quantum field in nD Minkowski space-time (cf. [2, Chap. 8.4]). Let

$$W_2(x) \equiv W_2(x; m^2, n) := (2\pi)^{-(n-2)/2} \mathscr{F}_{(k^0,\boldsymbol{k})}\Big(\frac{1}{2\omega(\boldsymbol{k})}\ \delta[k^0 - \omega(\boldsymbol{k})]\Big)(x), \tag{4.1}$$

where $\omega(\boldsymbol{k}) := \sqrt{m^2 + \boldsymbol{k}\cdot\boldsymbol{k}} \in \mathscr{C}^\infty(\mathbb{R}^{n-1})$. For $n \in 2\mathbb{N}$, $m \in \mathbb{R}^+$ the two-point Wightman distribution of the vector current is given by

$$\begin{aligned} &J^{\mu\nu}(x; m^2, n) \\ &:= \ -2^{n/2}\left\{2\big(\partial^\mu W_2 \underset{\mathrm{F}}{\circ} \partial^\nu W_2\big)(x) - \eta^{\mu\nu}\big(\partial^\rho W_2 \underset{\mathrm{F}}{\circ} \partial_\rho W_2\big)(x) + m^2\eta^{\mu\nu}\big(W_2 \underset{\mathrm{F}}{\circ} W_2\big)(x)\right\}. \end{aligned} \tag{4.2}$$

Here $(\eta^{\mu\nu}) := \mathrm{diag}(1, -1, ..., -1)$ and $\partial^\rho := \eta^{\rho\sigma}\partial_\sigma$ with $\partial_\rho := \frac{\partial}{\partial x^\rho}$. Note that the Fourier products on the right-hand side of (4.2) exist in the strong sense since the Fourier transforms of W_2 and $\partial^\mu W_2$ have their supports in the backward light cone $\overline{V}_- := \{k \in \mathbb{R}^n \mid -k^0 \geq |\boldsymbol{k}|\}$. We are interested in evaluating models with mass zero, i.e. we wish to take limits as $m \downarrow 0$.

Remark 4.1: For $n > 2$ the limit for $m \downarrow 0$ of the mass-parametrized distributions $W_2(x; m^2, n)$ exists in $\mathscr{S}'(\mathbb{R}^n)$. Therefore this limit can be performed for $J^{\mu\nu}(x; m^2, n)$, as given by (4.2) term by term: The continuity of the Fourier transform and of the convolution map (cf. Proposition 2.1) implies that the limit for $m \downarrow 0$ of each Fourier product is the Fourier product of the factors' limits.

The case $n = 2$ is the one where more difficult distributional products appear. In this case the computation is obstructed by the fact that $W_2(x; m^2, 2)$ does not have a limit in $\mathscr{D}'(\mathbb{R}^2)$ for $m \downarrow 0$.

Proposition 4.2: *For $n = 2$ define $W_2^0(x) := -\frac{1}{4\pi}\ln(-x^2 + i\epsilon x^0)\big|_{\epsilon\downarrow 0} \in \mathscr{S}'(\mathbb{R}^2)$. Here $x^2 := x \cdot x$. Then it holds:*

$$\begin{aligned} J^{\mu\nu}(x; m^2 = 0, 2) \ &:= \ \lim_{m\downarrow 0} J^{\mu\nu}(x; m^2, 2) \\ &= \ 2\left\{-2\big(\partial^\mu W_2^0 \underset{\mathrm{F}}{\circ} \partial^\nu W_2^0\big)(x) + \eta^{\mu\nu}\big(\partial^\rho W_2^0 \underset{\mathrm{F}}{\circ} \partial_\rho W_2^0\big)(x)\right\}. \end{aligned} \tag{4.3}$$

Proof: We verify below that

$$mW_2(x; m^2, 2) \to 0\ , \qquad \partial^\mu W_2(x; m^2, 2) \to \partial^\mu W_2^0(x) \tag{4.4}$$

as $m \downarrow 0$ in $\mathscr{S}'(\mathbb{R}^2)$. Equation (4.1) shows that the Fourier transforms of $W_2(x; m^2, 2)$ and $\partial^\mu W_2(x; m^2, 2)$ have their supports in the backward light cone. A continuity argument as in Remark 4.1, employing Proposition 2.1, implies that we can carry out the limit $m \downarrow 0$ in (4.2) term by term, which leads to (4.3).

To prove (4.4) we take the Fourier transform of $mW_2(x; m^2, 2)$. It equals $\frac{m}{2}[m^2 + (k^1)^2]^{-\frac{1}{2}}\ \delta\left(k^0 + \sqrt{m^2 + (k^1)^2}\right)$. But $m[m^2 + (k^1)^2]^{-\frac{1}{2}}$ is bounded from above by the locally integrable and at infinity polynomially bounded function $m(2m|k^1|)^{-\frac{1}{2}}$, thus $\mathscr{F}[mW_2(x; m^2, 2)]$ goes to zero as $m \downarrow 0$. Similarly,

$$\begin{aligned}
\mathscr{F}(\partial^1 W_2) &= \frac{ik^1}{2}\left[m^2+(k^1)^2\right]^{-\frac{1}{2}}\delta\left(k^0+\sqrt{m^2+(k^1)^2}\right) \\
&\qquad \longrightarrow -\frac{i}{2}\left\{\Theta(-k^1)\underset{\mathrm{p}}{\circ}\delta(k^0-k^1)-\Theta(k^1)\underset{\mathrm{p}}{\circ}\delta(k^0+k^1)\right\}, \\
\mathscr{F}(\partial^0 W_2) &= \frac{ik^0}{-2k^0}\,\delta\left(k^0+\sqrt{m^2+(k^1)^2}\right) \\
&\qquad \longrightarrow -\frac{i}{2}\left\{\Theta(-k^1)\underset{\mathrm{p}}{\circ}\delta(k^0-k^1)+\Theta(k^1)\underset{\mathrm{p}}{\circ}\delta(k^0+k^1)\right\},
\end{aligned}$$

as $m \downarrow 0$. On the other hand it is known [7, 14] that

$$\begin{aligned}
\mathscr{F}(W_2^0)(k) &= \left\{\frac{1}{(-k^0-k^1)_+}\otimes\delta(k^0-k^1)+\frac{1}{(-k^0+k^1)_+}\otimes\delta(k^0+k^1)+(C-\ln 2)\delta^2(k)\right\} \\
&= \frac{1}{(-2k^0)_+}\underset{\mathrm{p}}{\circ}\left[\delta(k^0-k^1)+\delta(k^0+k^1)\right]+(C-\ln 2)\delta^2(k).
\end{aligned} \tag{4.5}$$

Here $\frac{1}{(x)_+} := \frac{d}{dx}\{\Theta(x)\ln x\} \in \mathscr{S}'(\mathbb{R})$ and C denotes Euler's constant. The last equality in equation (4.5) follows from the identity

$$\frac{1}{(-k^0-\alpha k^1)_+}\otimes\delta(k^0-\alpha k^1)=\frac{1}{(-2k^0)_+}\underset{\mathrm{p}}{\circ}\delta(k^0-\alpha k^1)$$

for $\alpha \neq 0$. Finally, multiplying (4.5) by ik^1 and ik^0, using (2.8), results in the second assertion of (4.4). ∎

Our next task is to compute $J^{\mu\nu}(x; m^2=0, 2)$ further. We interprete the factors of the products which appear in equation (4.3) as x^1-parametrized distributions in $\mathscr{S}'(\mathbb{R})$ and then employ Proposition 3.2.

Proposition 4.3: *The two-point Wightman distribution of the vector current for a free, zero-mass Dirac quantum field in 2D space-time is given by*

$$J^{\mu\nu}(x; m^2=0, n=2) = -\frac{1}{\pi}\,\partial^\mu\partial^\nu W_2^0(x) = (2\pi)^{-2}\partial^\mu\partial^\nu \ln(-x^2+i\epsilon x^0)\big|_{\epsilon\downarrow 0}. \tag{4.6}$$

Proof: First we note the following identities for W_2^0:

$$\begin{aligned}
W_2^0(x) &= -\frac{1}{4\pi}\left\{\ln|x^2|+i\pi\,\Theta(x^2)\,\mathrm{sign}\,x^0\right\} \\
&= -\frac{1}{4\pi}\left\{\ln(x^0-x^1-i\epsilon)\big|_{\epsilon\downarrow 0}+\ln(x^0+x^1-i\epsilon)\big|_{\epsilon\downarrow 0}+i\pi\right\}.
\end{aligned} \tag{4.7}$$

From (4.7) we infer that

$$\partial^\rho W_2^0(x) = -\frac{1}{4\pi}\left\{(x^0-x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}+(-1)^\rho\,(x^0+x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right\}, \tag{4.8}$$

$$\partial^\rho\partial^\sigma W_2^0(x) = \frac{1}{4\pi}\left\{(x^0-x^1-i\epsilon)^{-2}\big|_{\epsilon\downarrow 0}+(-1)^{\rho+\sigma}\,(x^0+x^1-i\epsilon)^{-2}\big|_{\epsilon\downarrow 0}\right\}. \tag{4.9}$$

These formulae show that the distributions on the right-hand side of (4.8) and (4.9) can be viewed as smooth $\mathscr{S}'(\mathbb{R})$-valued functions, where x^0 as well as x^1 may serve as parameter. Recall now the fact that the Fourier transform of the distribution $(x-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0} = \delta_-(x) \in \mathscr{S}'(\mathbb{R})$ has its support equal to the half-line $(-\infty, 0] =: \mathbb{R}_0^-$. Then by Proposition 3.2 it follows that the Fourier products of $(x^0+\alpha x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}$ and $(x^0+\beta x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}$ considered as distributions in $\mathscr{S}'(\mathbb{R}^2)$ exist for all $\alpha, \beta \in \mathbb{R}$ and coincide with the associated distributions in $\mathscr{D}'(\mathbb{R}^2)$ of their one-dimensional Fourier products for x^1 fixed:

$$\left[(x^0+\alpha x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \underset{\mathrm{F}_{(x^0,x^1)}}{\circ} \left[(x^0+\beta x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right]$$
$$= \iota \left\{ \left[(x^0+\alpha x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \underset{\mathrm{F}_{x^0}}{\circ} \left[(x^0+\beta x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \right\}.$$

Furthermore, if $\alpha = \beta$, we have on account of formulae (2.9) and (2.7)

$$\delta_-(x^0+\alpha x^1) \underset{\mathrm{F}_{x^0}}{\circ} \delta_-(x^0+\alpha x^1) = \big(\delta_- \underset{\mathrm{F}}{\circ} \delta_-\big)(x^0+\alpha x^1) = (x^0+\alpha x^1-i\epsilon)^{-2}\big|_{\epsilon\downarrow 0}.$$

Therefore the product $\partial^\mu W_2^0 \underset{\mathrm{F}}{\circ} \partial^\nu W_2^0$ can be calculated term by term and one obtains

$$\begin{aligned} &\big(\partial^\mu W_2^0 \underset{\mathrm{F}}{\circ} \partial^\nu W_2^0\big)(x) \\ &= \frac{1}{(4\pi)^2}\Big\{ (x^0-x^1-i\epsilon)^{-2}\big|_{\epsilon\downarrow 0} + (-1)^{\mu+\nu}\,(x^0+x^1-i\epsilon)^{-2}\big|_{\epsilon\downarrow 0} \\ &\quad + [(-1)^\mu+(-1)^\nu]\left[(x^0-x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \underset{\mathrm{F}}{\circ} \left[(x^0+x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right]\Big\} \\ &= \frac{1}{4\pi}\,\partial^\mu\partial^\nu W_2^0(x) + \eta^{\mu\nu}\frac{1}{8\pi^2}\left[(x^0-x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \underset{\mathrm{F}}{\circ} \left[(x^0+x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right]. \end{aligned} \tag{4.10}$$

The second equality is due to (4.9) and the identity $(-1)^\mu + (-1)^\nu = 2\eta^{\mu\nu}$. From this it follows immediately that

$$\begin{aligned} &\big(\partial^\rho W_2^0 \underset{\mathrm{F}}{\circ} \partial_\rho W_2^0\big)(x) \\ &= \frac{1}{4\pi}\,\Box W_2^0(x) + \frac{1}{4\pi^2}\left[(x^0-x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right] \underset{\mathrm{F}}{\circ} \left[(x^0+x^1-i\epsilon)^{-1}\big|_{\epsilon\downarrow 0}\right]. \end{aligned}$$

Here $\Box := \partial^\rho\partial_\rho$. Inserting these equations into (4.3) and using the fact that $\Box W_2^0 = 0$, we finally get the result: $J^{\mu\nu}(x; m^2 = 0, 2) = -\frac{1}{\pi}\,\partial^\mu\partial^\nu W_2^0(x)$. ∎

Remark 4.4: The second term in formula (4.10) may be written as $\delta_-(x^0-x^1) \underset{\mathrm{F}}{\circ} \delta_-(x^0+x^1)$. This product exists at every fixed x^1. On the other hand, we could not have taken x^0 as parameter, because the corresponding product would not exist at $x^0 = 0$.

Remark 4.5: Equation (4.6) has been obtained in a formal way, i.e. without specification of the distribution products, already in [23, equations (4.42) and (4.43)]. It presents the simplest example for the occurrence of bosonic one particle sectors in purely fermionic quantum field models. In view of this bosonization phenomenon's importance [9], it appears reasonable to establish equation (4.6) on as firm grounds as possible.

Our next application concerns the time ordered vacuum expectation value of the vector current for a free, zero-mass Dirac quantum field in 2D space-time. It is defined by

$$TJ^{\mu\nu}(x; m^2 = 0, n = 2) := \Theta(x^0) \circ J^{\mu\nu}(x; 0, 2) + \Theta(-x^0) \circ J^{\nu\mu}(-x; 0, 2). \tag{4.11}$$

We note that by formulae (4.6) and (4.9), both $W_2^0(x)$ and $J^{\mu\nu}(x; 0, 2)$ can be viewed as smooth $\mathscr{S}'(\mathbb{R})$-valued functions with x^0 as parameter. In particular, the products with the Heaviside function in (4.11) have a meaning as parameter products. Alternatively, they can be interpreted as Fourier products according to Proposition 3.1. The result is computed as follows.

Proposition 4.6: *The distribution $TJ^{\mu\nu}$ defined by equation (4.11) is given by*

$$TJ^{\mu\nu}(x; m^2 = 0, n = 2) = \frac{i}{\pi}\left\{\partial^\mu\partial^\nu\Delta_F^0(x) - \eta_0^\mu\eta_0^\nu\delta^2(x)\right\} \in \mathscr{S}'(\mathbb{R}^2). \tag{4.12}$$

Here

$$\begin{aligned}\Delta_F^0(x) &:= i\left\{\Theta(x^0)\circ W_2^0(x) + \Theta(-x^0)\circ W_2^0(-x)\right\} \\ &= -\frac{i}{4\pi}\ln(-x^2 + i\epsilon)\big|_{\epsilon\downarrow 0} \in \mathscr{S}'(\mathbb{R}^2)\end{aligned} \tag{4.13}$$

denotes the Feynman propagator associated with W_2^0 and $\eta_0^\mu := \eta^{\mu 0}$.

Proof: The proof is based on the parameter product interpretation of equation (4.11). Formulae (3.3) and (3.4) yield

$$\begin{aligned}&\partial^\mu\partial^\nu\left\{\Theta(x^0)\underset{\mathrm{p}}{\circ}W_2^0(x)\right\} \\ &\quad = \Theta(x^0)\underset{\mathrm{p}}{\circ}\partial^\mu\partial^\nu W_2^0(x) + \eta_0^\mu\partial^\nu\left\{\delta(x^0)\underset{\mathrm{p}}{\circ}W_2^0(x)\right\} + \eta_0^\nu\delta(x^0)\underset{\mathrm{p}}{\circ}\partial^\mu W_2^0(x).\end{aligned}$$

In terms of the commutator distribution $\Delta^0(x) := -i\{W_2^0(x) - W_2^0(-x)\}$ this leads to

$$\begin{aligned}&\Theta(x^0)\underset{\mathrm{p}}{\circ}\partial^\mu\partial^\nu W_2^0(x) + \Theta(-x^0)\underset{\mathrm{p}}{\circ}\partial^\mu\partial^\nu W_2^0(-x) \\ &\quad = -i\partial^\mu\partial^\nu\Delta_F^0(x) - i\eta_0^\mu\partial^\nu\left\{\delta(x^0)\underset{\mathrm{p}}{\circ}\Delta^0(x)\right\} - i\eta_0^\nu\delta(x^0)\underset{\mathrm{p}}{\circ}\partial^\mu\Delta^0(x).\end{aligned}$$

Now we compute the parameter products of $\delta(x^0)$ with Δ^0 and $\partial^\mu\Delta^0$. From equation (4.7) we get for the commutator distribution and its derivative

$$\begin{aligned}\Delta^0(x) &= -\frac{1}{2}\Theta(x^2)\,\mathrm{sign}\,x^0 = -\frac{1}{4}\left\{\mathrm{sign}(x^0 - x^1) + \mathrm{sign}(x^0 + x^1)\right\}\ , \\ \partial^\mu\Delta^0(x) &= -\frac{1}{2}\left\{\delta(x^0 - x^1) + (-1)^\mu\delta(x^0 + x^1)\right\}\ .\end{aligned}$$

These identities make it obvious that $\delta(x^0)\underset{\mathrm{p}}{\circ}\Delta^0(x) = 0$ and $\delta(x^0)\underset{\mathrm{p}}{\circ}\partial^\mu\Delta^0(x) = -\eta_0^\mu\delta^2(x)$ hold. Thus due to (4.6) the proof of equation (4.12) is completed. The explicit expression (4.13) for the Feynman propagator is obtained as follows: Let $\varphi \in \mathscr{D}(\mathbb{R}^2)$. Then it holds

$$\begin{aligned}\langle\Delta_F^0, \varphi\rangle &= i\Big\langle\Theta(x^0), \big\langle W_2^0(x^0,\cdot), \{\varphi(x^0,\cdot) + \check\varphi(x^0,\cdot)\}\big\rangle\Big\rangle \\ &= -\frac{i}{4\pi}\int_0^\infty dx^0\ \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}} dx^1 \ln(-x^2 + i\epsilon x^0)\ \{\varphi(x) + \varphi(-x)\} \\ &= -\frac{i}{4\pi}\int_{\mathbb{R}} dx^0\ \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}} dx^1 \ln(-x^2 + i\epsilon)\ \varphi(x) \\ &= -\frac{i}{4\pi}\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^2} d^2x\ \ln(-x^2 + i\epsilon)\ \varphi(x) \\ &= \Big\langle -\frac{i}{4\pi}\ln(-x^2 + i\epsilon)\big|_{\epsilon\downarrow 0}, \varphi(x)\Big\rangle\ .\end{aligned}$$

Lebesgue's dominated convergence theorem allows to interchange the integration with the limit $\epsilon \downarrow 0$. ∎

Remark 4.7: The Fourier products arising in (4.11) and Proposition 4.6 exist actually in the strong sense. In fact, the Fourier product of any tempered distribution $V(x^0)\otimes 1(x^1) \in \mathscr{S}'(\mathbb{R}^2)$ and of $W_2^0(x)$, and hence also its derivatives, exists in the strong sense: the supports of their Fourier transforms are in favorable position. To see this, we recall the Fourier transform of $W_2^0(x)$. By equation (4.5) we have that $\mathscr{F}(W_2^0)(k) = \tilde W_-(k) + \tilde W_+(k)$ with

$$\tilde W_\pm(k) := \frac{1}{(-k^0 \pm k^1)_+}\otimes\delta(k^0 \pm k^1) + \frac{1}{2}\ (C - \ln 2)\ \delta^2(k)\ .$$

The supports of the distributions $\tilde{W}_\pm(k)$ are subsets of the closed acute cones $\Gamma_\pm := \{k \in \mathbb{R}^2 | k \cdot k \leq 0, \pm k^1 \geq 0\}$ with axial vectors $N_\pm := (0, \pm 1)$:

$$\operatorname{supp} \tilde{W}_\pm = \left\{ k \in \mathbb{R}^2 \mid k^0 = \mp k^1 \leq 0 \right\} \subset \Gamma_\pm \ .$$

Since the Fourier transform of $V(x^0)$ viewed as a two-dimensional tempered distribution is $\mathscr{F}[V(x^0) \otimes 1(x^1)](k) = (2\pi)^{\frac{1}{2}} (\mathscr{F}V)(k^0) \otimes \delta(k^1)$, we conclude that its support $\operatorname{supp} \mathscr{F}(V \otimes 1) \subset \mathbb{R} \times \{0\}$ is contained in each of the two closed half-spaces $\mathbb{R} \times \mathbb{R}_0^\pm$ with $N_\pm$ as their interior normals. Therefore, the supports of $\mathscr{F}(V \otimes 1)$ and $\tilde{W}_\pm$ are in favorable position so that their $\mathscr{S}'$-convolution exists.

Accordingly, one could compute the products in (4.11) and (4.13) by means of convolving the occurring Fourier transforms. This, however, would require computations more complicated than ours.

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