Fourier Representation for the Two-Point Function of the Two-Dimensional Massless Scalar Field

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Abstract. The Fourier transform of the (indefinite metric) Wightman two-point function $-(1/4\pi)\ln(-x^2+i\epsilon x^0)|_{\epsilon\downarrow 0}$ of a free massless scalar quantum field in two-dimensional spacetime has been inconsistently reported by various authors. We compute the correct one from the definition of the Fourier transform of tempered distributions.

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The set of elements of $\mathscr{S}'(\mathbb{R}^2)$ which are invariant under orthochronous Lorentz transformations and have their supports on the forward light cone reads $\mathbb{C}\omega(p) + \mathbb{C}\delta^2(p)$ [1]. Here

$$\omega(p) := \left\{ \frac{1}{(p^0+p^1)_+} \otimes \delta(p^0-p^1) + \frac{1}{(p^0-p^1)_+} \otimes \delta(p^0+p^1) \right\} \in \mathscr{S}'(\mathbb{R}^2), \quad (1)$$

$$\frac{1}{(x)_+} := \frac{\mathrm{d}}{\mathrm{d}x}\{\theta(x)\ln(x)\} \in \mathscr{S}'(\mathbb{R}).$$

This fact causes ω to be one of the cornerstones for the discussion of two-dimensional quantum field theoretic models involving a free zero mass scalar quantum field [2]. Nevertheless, concerning the Fourier transform of ω, formally defined by

$$(\mathscr{F}\omega)(x) := \int \mathrm{d}^2p\ \omega(p)\ \frac{\mathrm{e}^{-ip\cdot x}}{2\pi}\ ,$$

a general agreement on some of its details is lacking. Although it is well known that

$$(\mathscr{F}\omega)(x) = -\frac{a}{4\pi}\ln(-x^2+i\epsilon x^0)\big|_{\epsilon\downarrow 0} + b\ , \quad x^2 := (x^0)^2 - (x^1)^2\ ,$$

the values of the constants a and b are inconsistent, as they are stated (more or less explicitly) in the literature. We have found the following values ($C = -\Gamma'(1)$ denotes Euler's constant):

(1) Ref. [1]: $a = 2$, $b = -C/\pi$,

(2) Ref. [3]: $a = 1$, $b = -(\ln 2 - C)/8\pi^2$,

(3) Ref. [4]: $a = 1/2\pi$, $b = 0$,

(4) Ref. [5]: $a = 1$, $b = (\ln 2 - C)/2\pi$.

Although the precise values of a and b are irrelevant for many problems, the conflicting statements (1) to (4) may lead to unnecessary confusion. In order to resolve this question, we present a rigorous computation of $\mathscr{F}\omega$, which demonstrates that statement (4) is correct. Our notation is as follows. We shall have to deal with the Fourier transform of $T \in \mathscr{S}'(\mathbb{R}^n)$ for the cases $n = 1, 2$:

$$\langle \mathscr{F}T,\ f\rangle := \langle T,\ \mathscr{F}f\rangle, \quad \mathscr{F}f(p) := \int_{\mathbb{R}^n} \mathrm{d}^n x\ f(x)\ \frac{\mathrm{e}^{-ip\cdot x}}{(2\pi)^{n/2}},\ \forall f \in \mathscr{S}(\mathbb{R}^n),$$

with $p \cdot x := p^0x^0 - p^1x^1$ for $n = 2$, and $p \cdot x := px$ for $n = 1$.

We now turn to the computation of $\mathscr{F}\omega$. First note that for $\alpha \in \mathbb{R} \setminus \{0\}$, the following identity holds:

$$\frac{1}{(p^0 + \alpha p^1)_+} \otimes \delta(p^0 - \alpha p^1) = A\left[\frac{1}{(p^0)_+} \otimes \delta(p^1)\right], \quad \text{with } A^{-1} = \begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \end{pmatrix}.$$

Here the composition of a distribution $U \in \mathscr{S}'(\mathbb{R}^2)$ and a nonsingular linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as: $\langle AU,\ \varphi\rangle := \langle U,\ |\det A| A^{-1}\varphi\rangle$, with $(A^{-1}\varphi)(x) := \varphi(Ax)$ for $\varphi \in \mathscr{S}(\mathbb{R}^2)$. One immediately checks $\mathscr{F}(AU) = |\det A|(A^*)^{-1}(\mathscr{F}U)$, where A^* denotes the adjoint linear map of A with respect to the Minkowski metric, i.e. $A^* = \eta A^T \eta$ with $\eta := \mathrm{diag}(1, -1)$. Therefore, we conclude that

$$\mathscr{F}\left[\frac{1}{(p^0 + \alpha p^1)_+} \otimes \delta(p^0 - \alpha p^1)\right] = |\det A|(A^*)^{-1}\mathscr{F}\left[\frac{1}{(p^0)_+} \otimes \delta(p^1)\right]. \qquad (2)$$

We now need the one-dimensional Fourier transform of $1/(p^0)_+$.

LEMMA. *The Fourier transform of* $1/(q)_+ \in \mathscr{S}'(\mathbb{R})$ *is given by*

$$\mathscr{F}\left[\frac{1}{(q)_+}\right] = -(2\pi)^{-1/2}\left\{\ln(x - i\epsilon)|_{\epsilon\downarrow 0} + \frac{i\pi}{2} + C\right\}, \quad C := -\Gamma'(1).$$

Proof. Let $g \in \mathscr{S}(\mathbb{R})$, then

$$\begin{aligned}\left\langle \mathscr{F}\frac{1}{(q)_+},\ g\right\rangle &:= \left\langle \frac{1}{(q)_+},\ \mathscr{F}g\right\rangle \\ &= \lim_{\epsilon\downarrow 0}\left\{\int_\epsilon^\infty \frac{\mathrm{d}q}{q}\ (\mathscr{F}g)(q) + \ln\epsilon\ (\mathscr{F}g)(0)\right\} \\ &= (2\pi)^{-1/2}\lim_{\epsilon\downarrow 0}\left\{\lim_{n\to\infty}\int_\epsilon^n \frac{\mathrm{d}q}{q}\int_{\mathbb{R}} \mathrm{d}x\ \mathrm{e}^{-iqx}g(x) + \ln\epsilon\int_{\mathbb{R}} \mathrm{d}x\ g(x)\right\}.\end{aligned}$$

Since

$$\int_\epsilon^n \frac{\mathrm{d}q}{q}\int_{\mathbb{R}} \mathrm{d}x\left|\mathrm{e}^{-iqx}g(x)\right| < \infty,$$

one can interchange the order of integration by using Fubini's theorem. Observe now that for $\epsilon > 0$ and $x \neq 0$

$$\begin{aligned}
&\left| \int_\epsilon^n \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iqx} \right| \\
&\quad \leqslant \left| \int_{\epsilon|x|}^1 \frac{\mathrm{d}q}{q} \right| + \left| \int_1^{n|x|} \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iq} \right| \\
&\quad \leqslant |\ln(\epsilon|x|)| + 2 + |\ln|x|| \leqslant 2 + 2|\ln(|x|)| + |\ln\epsilon| \ , \quad \forall n \geqslant 1,
\end{aligned}$$

on account of

$$\left| \int_1^{n|x|} \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iq} \right| \leqslant \left| \frac{i\ \mathrm{e}^{-iq}}{q} \Big|_1^{n|x|} \right| + \int_1^{n|x|} \frac{\mathrm{d}q}{q^2} \leqslant 2 \ , \quad \text{for } n|x| \geqslant 1$$

and

$$\left| \int_1^{n|x|} \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iq} \right| \leqslant \int_{|x|}^1 \frac{\mathrm{d}q}{q} = |\ln|x||, \quad \text{for } n|x| \leqslant 1,\ n \geqslant 1.$$

This bound is a locally integrable and, at infinity, polynomially bounded function of x. Thus, by Lebesgue's dominated convergence theorem, it is possible to interchange the x-integration with performing the limit $n \to \infty$:

$$\left\langle \mathscr{F} \frac{1}{(q)_+},\ g \right\rangle = (2\pi)^{-1/2} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \mathrm{d}x\ g(x) \left\{ \int_\epsilon^\infty \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iqx} + \ln\epsilon \right\}.$$

For $x \neq 0$, it is straightforward to check

$$\begin{aligned}
&\int_\epsilon^\infty \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iqx} + \ln\epsilon \\
&\quad = -\ln|x| + \ln(\epsilon|x|) - \mathrm{ci}(\epsilon|x|) + i\,\mathrm{sign}(x)\,\mathrm{si}(\epsilon|x|) \\
&\quad = -\ln|x| + i\,\mathrm{sign}(x)\,\mathrm{si}(\epsilon|x|) - C + \int_0^{\epsilon|x|} \mathrm{d}q\ \frac{1-\cos(q)}{q} \ ,
\end{aligned}$$

where $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ denote the sine integral and cosine integral, respectively, as defined in [6]. The last equality is due to formula 8.230 2. of [6]. Since

$$\frac{1-\cos(x)}{x} \leqslant 1 \quad \text{and} \quad |\,\mathrm{si}(x)| \leqslant \frac{\pi}{2}, \quad \forall x \geqslant 0,$$

one obtains the ϵ-independent bound

$$\left| \int_\epsilon^\infty \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iqx} + \ln\epsilon \right| \leqslant \frac{\pi}{2} + C + |\ln|x|| + |x| \ , \quad \text{for } 0 < \epsilon \leqslant 1,$$

and, with $\mathrm{si}(0) = -(\pi/2)$,

$$\begin{aligned} \lim_{\epsilon\downarrow 0} & \left\{ \int_\epsilon^\infty \frac{\mathrm{d}q}{q}\, \mathrm{e}^{-iqx} + \ln\epsilon \right\} \\ &= -\ln|x| - \frac{i\pi}{2}\,\mathrm{sign}(x) - C \\ &= -\left\{ \ln(x - i\epsilon)|_{\epsilon\downarrow 0} + \frac{i\pi}{2} + C \right\}. \end{aligned}$$

Thus, Lebesgue's dominated convergence theorem allows the interchange of the x-integration with performing the limit $\epsilon \downarrow 0$. This completes the proof. □

By means of this Lemma we obtain

$$\mathscr{F}\left[\frac{1}{(p^0)_+} \otimes \delta(p^1)\right](x) = -\frac{1}{2\pi}\left\{\ln(x^0 - i\epsilon)|_{\epsilon\downarrow 0} \otimes 1(x^1) + \frac{i\pi}{2} + C\right\},$$

with $1(x^1) = 1$, $\forall x^1 \in \mathbb{R}$. Inserting this into Equation (2), and using the identities

$$A^* = \frac{1}{2}\begin{pmatrix} 1 & -\alpha^{-1} \\ -1 & -\alpha^{-1} \end{pmatrix} \quad \text{and} \quad |\det A| = (2|\alpha|)^{-1}$$

yields

$$\begin{aligned} \mathscr{F} & \left[\frac{1}{(p^0 + \alpha p^1)_+} \otimes \delta(p^0 - \alpha p^1)\right](x) \\ &= -\frac{1}{4\pi|\alpha|}\left\{\ln\left(x^0 - \frac{x^1}{\alpha} - i\epsilon\right)\Big|_{\epsilon\downarrow 0} - \ln 2 + \frac{i\pi}{2} + C\right\}. \end{aligned}$$

Now, according to Equation (1), by summing up this result for $\alpha = \pm 1$, we finally get for the Fourier transform of ω:

$$\begin{aligned} \mathscr{F}(\omega)(x) &= -\frac{1}{4\pi}\left\{\ln(x^0 - x^1 - i\epsilon)|_{\epsilon\downarrow 0} + \ln(x^0 + x^1 - i\epsilon)|_{\epsilon\downarrow 0} + i\pi\right\} + \\ & \quad + \frac{1}{2\pi}(\ln 2 - C) \\ &= -\frac{1}{4\pi}\ln(-x^2 + i\epsilon x^0)|_{\epsilon\downarrow 0} + \frac{1}{2\pi}(\ln 2 - C). \end{aligned}$$

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