Hypoplasticity for beginners.

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Abstract

There are a lot of constitutive laws to describe the deformation behaviour of soil. The hypoplastic law is a good choice for cohesionless soils. It describes the behaviour of soil very realistic, i.e. non-linear and inelastic. It is formulated for the general three-dimensional case. Therefore the equations are not quite simple. This scares many potential users out of utilising hypoplasticity.

Some fundamental ideas of the hypoplastic material law are shown with simple one dimensional examples. Thus the major functionality of this law becomes clear.

1 Introduction, definitions

Constitutive laws describe the deformation behaviour of materials. They are mathematical formulations of the stress-strain relation.

Here two simple one-dimensional constitutive laws are developed. One to simulate the behaviour of a soil sample in the confined compression test, an the other to simulate the triaxial test. Both constitutive laws are of the form of the hypoplastic law.

Constitutive law developers define the signs for stress and strain as in the general mechanics. Thus compression stresses and strains are negative, contrary to the usual practice in soil mechanics. We comply with this convention.

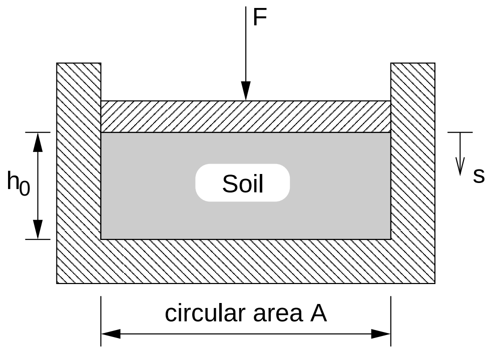

Figure 1: Confined compression test: schematic experimental setup

2 Confined compression test

The cylindrical soil sample in fig. 1 is axially compressed with confined lateral strains. The vertical stress $\sigma = -F/A$ as function of the vertical strain $\varepsilon = -s/h_0$ is plotted in fig. 2 for loading and unloading of a sand sample.

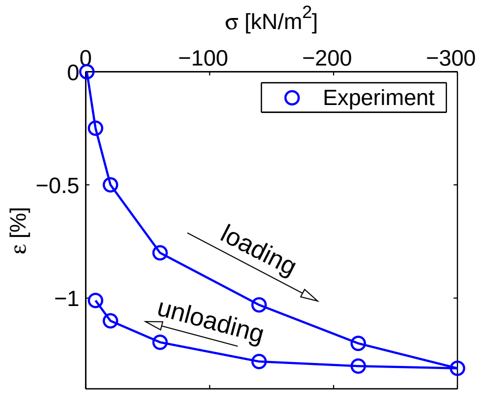

Figure 2: Confined compression test with loose sand: stress-strain relation

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The curved lines indicate a non-linear behaviour. The different branches for loading and unloading signify inelastic behaviour. Our goal is to find a mathematical formulation as simple as possible, but representing a good approximation to this behaviour however.

2.1 Most simple model

The most simple approximation of the stress-strain curve of a confined compression test are two straight lines (fig. 3).

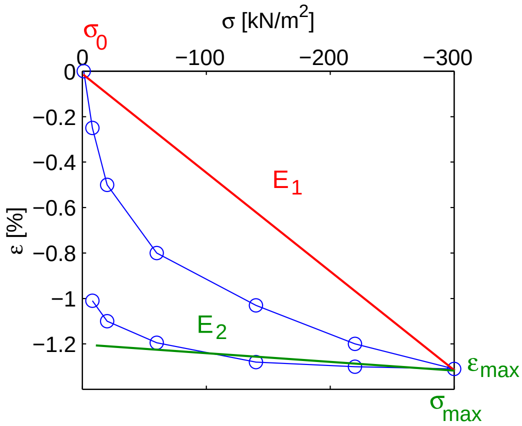

Figure 3: Linear inelastic approximation

This is called a linear inelastic constitutive law. A naive way to formulate this law is a direct function between stress and strain:

$$\text{loading:} \quad \sigma = \sigma_0 + E_1\varepsilon \qquad (1)$$

$$\text{unloading:} \quad \sigma = \sigma_{max} + E_2(\varepsilon - \varepsilon_{max}) \qquad (2)$$

But the maximum stress σ_{max} is usually not known a priori, it is a result of the loading history. Thus the equation for unloading (2) is defined not very convenient. In addition we cannot decide whether there is loading or unloading alone with the current value of the strain. We have to regard the change of the strain anyway. This leads directly to the following incremental formulation, which is based on the strain rate.

A more clever formulation of the non-linear inelastic behaviour in fig. 2 is a rate equation. Thinking of loading as a process in time, we introduce the parameter time t. Now the change of the stress and the strain are regarded as function of time for a given strain rate $\dot\varepsilon(t)$. The strain rate $\dot\varepsilon(t)$ is the derivative of the strain with respect to the time $\dot\varepsilon(t) = \mathrm{d}\varepsilon(t)/\mathrm{d}t$. The strain rate is negative for loading (compression) $\dot\varepsilon < 0$ and positive for unloading (expansion) $\dot\varepsilon > 0$.

A simple constitutive law of the rate type to simulate the inelastic behaviour looks then like:

$$\text{loading } \dot\varepsilon < 0: \quad \dot\sigma = E_1\dot\varepsilon \qquad (3)$$

$$\text{unloading } \dot\varepsilon > 0: \quad \dot\sigma = E_2\dot\varepsilon \qquad (4)$$

Hypoplasticity is a constitutive law of the rate type. It is a relation which associates the strain rate to the stress rate.

In order to obtain the stress-strain relation, we have to integrate this rate equation over time. This is done by the famous *time integration* of the constitutive law.

Time integration of (3) for loading yields:

$$\int_0^t \dot\sigma\,\mathrm{d}t = \int_0^t E_1\dot\varepsilon\,\mathrm{d}t$$

$$\sigma(t) - \sigma(0) = E_1\varepsilon(t) - E_1\varepsilon(0)$$

The initial values are $\varepsilon(0) = 0$ and $\sigma(0) = \sigma_0$. Thus stress as function of strain during loading is

$$\sigma = \sigma_0 + E_1\varepsilon \ ,$$

which is exactly the same as (1). If loading is changed to unloading at time t_1, time integration of (4) with the initial values $\varepsilon(t_1) = \varepsilon_{max}$ and $\sigma(t_1) = \sigma_{max}$ gives

$$\sigma = \sigma_{max} + E_2(\varepsilon - \varepsilon_{max}) \ ,$$

which is the unloading line (2).

We can combine equations (3) and (4) to

$$\dot\sigma = \frac{E_1 + E_2}{2}\dot\varepsilon + \frac{E_2 - E_1}{2}|\dot\varepsilon| \ . \qquad (5)$$

The absolute value of the strain rate provides different stiffnesses for loading and unloading.

The inelastic behaviour of the hypoplastic constitutive law is modeled by using the modulus of the strain rate.

2.2 Improved model

Since the linear stress-strain relation is rather unsatisfactory, we try now to get a curved line. The stiffness of soil is often assumed to be proportional to the stress. We can find a mathematical formulation, which is able to simulate this, by multiplying the right hand side of (5) by the stress and introducing two new constants

$$\dot{\sigma} = C_1 \sigma \dot{\varepsilon} + C_2 \sigma |\dot{\varepsilon}| \ . \tag{6}$$

Time integration for loading ($\dot{\varepsilon} < 0$) yields

$$\ln \frac{\sigma}{\sigma_0} = (C_1 - C_2)(\varepsilon - \varepsilon_0) \ ,$$

the well-known logarithmic stress-strain relation in the confined compression test. These curves are plotted for loading and unloading in fig. 4.

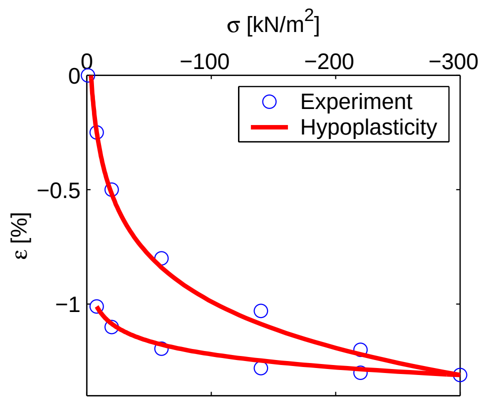

Figure 4: Hypoplastic approximation with (6), $C_1 = -775$, $C_2 = -433$ ($\sigma_0 = -3,4$ kN/m^2)

2.3 Connection to traditional soil mechanics

Loading in the confined compression test is usually approximated by

$$d\sigma = -C_c^{-1} \sigma de \ ,$$

wherein C_c is the compression index.

The void ratio e in the confined compression test can be expressed as function of the strain $e = e_0 + (1 + e_0)\varepsilon$, with the initial void ratio e_0 at the beginning of the compression. The differential of the void ratio is $de = (1 + e_0) d\varepsilon$ and for the stress applies

$$d\sigma = -\frac{1 + e_0}{C_c} \sigma d\varepsilon \ .$$

We rewrite (6) for compression ($\dot{\varepsilon} < 0$)

$$\dot{\sigma} = (C_1 - C_2) \sigma \dot{\varepsilon} \ ,$$

and compare it with the above equation. We find out that

$$-\frac{1 + e_0}{C_c} = C_1 - C_2 \ . \tag{7}$$

For unloading similar applies

$$-\frac{1 + e_0}{C_e} = C_1 + C_2 \ ,$$

where C_e is the expansion index.

The simplified hypoplasticity law (6) corresponds to the conventional soil deformation behaviour concept. The constants can be determined by:

$$\begin{aligned} C_1 &= -\frac{1 + e_0}{2} \frac{C_e + C_c}{C_e C_c} \\ C_2 &= -\frac{1 + e_0}{2} \frac{C_c - C_e}{C_e C_c} \end{aligned}$$

The constraint modulus of soil is usually assumed to be proportional to the stress

$$E_s = -(1 + e_0)/C_c \sigma \ .$$

We see with the help of (7) that $E_s = (C_1 - C_2)\sigma$ for loading. Thus the factors $C_1 \sigma$ and $C_2 \sigma$ in (6) denote a stiffness.[1]

Remark: Alternatively we can deduce this mathematically. The tangent constraint modulus is defined as $d\sigma / d\varepsilon$. The stress is now given as function of time. We use the chain rule to show

$$\frac{d\sigma}{d\varepsilon} = \frac{d\sigma}{dt} \frac{dt}{d\varepsilon} = \frac{\frac{d\sigma}{dt}}{\frac{d\varepsilon}{dt}} = \frac{\dot{\sigma}}{\dot{\varepsilon}} \ . \tag{8}$$

With this we work out the loading constraint modulus from (6)

$$\frac{d\sigma}{d\varepsilon} = \frac{\dot{\sigma}}{\dot{\varepsilon}} = (C_1 - C_2)\sigma \ .$$

[1]Note: C_1 and C_2 are negative, but the stress σ is negative too. Thus the stiffness becomes positive.

The stress rate in (6) depends linearly on the stress. That causes the proportionality of the stiffness to the stress.

Remark: Mathematically a function is called homogeneous of the first degree, if $y = f(\lambda x) = \lambda f(x)$ is valid for all λ. As we see easily the hypoplastic law (6) has this property for the argument stress. If we postulate the proportionality of the stiffness to the stress, we come out with a constitutive law that is homogeneous of the first degree in σ, which means that the stress terms appear only to the first power.

The non-linear behaviour of the hypoplastic constitutive law is modeled by the stress dependence of the stiffness.

2.4 Rate-independence

As a first approximation the stress-strain relation of cohesionless soil is independent of the loading rate. This should be considered by the constitutive law.

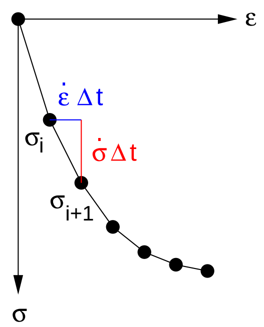

Figure 5: Numerical time integration of the constitutive law

In fig. 5 we see the result of a numerical time integration for a time step Δt. Rate-independence means that the gradient of the curve (stiffness) $\frac{\Delta\sigma}{\Delta\varepsilon} = \frac{\dot\sigma\Delta t}{\dot\varepsilon\Delta t}$ does not depend on the strain rate $\dot\varepsilon$. Thus the stress rate $\dot\sigma$ must be precisely twice as large for a double strain rate. Therefore the constitutive law may not contain terms like e.g. $\dot\varepsilon^2$.

Remark: However, if the strain rate changes the sign, the stiffness has to change! Therefore hypoplasticity is *positively* homogeneous of the first degree in strain rate, i.e. $y = f(\lambda x) = \lambda f(x)$ applies only to positive λ).

Hypoplastic constitutive are positively homogeneous of the first degree in the strain rate.

3 Triaxial-Test

In the triaxial test a cylindrical soil sample is axially compressed by a vertical stress σ_1 with constant lateral stress σ_2 (fig. 6).

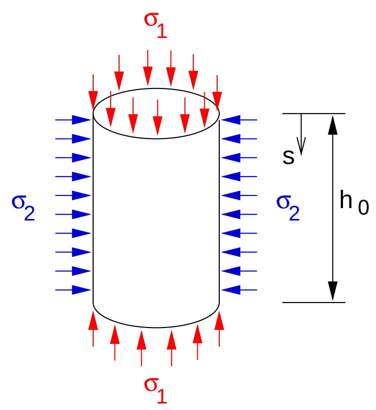

Figure 6: Triaxial test, schematically

The qualitatively relation between the vertical stress σ_1 and the vertical strain $\varepsilon_1 = -s/h_0$ is plotted in fig. 7.

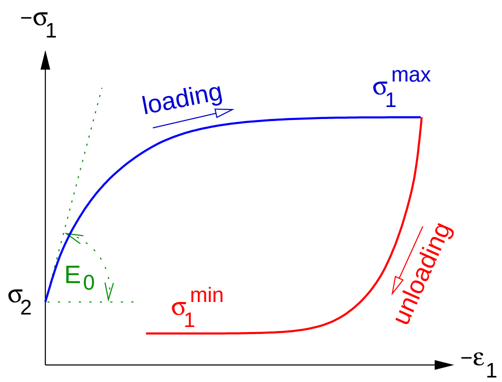

Figure 7: Vertical stress in triaxial test

Now we try to design a simple one-dimensional constitutive law, with the main properties of a hypoplastic law, which were discussed in the previous section.

The constitutive law has to fulfil three requirements:

1. different stiffness for loading and unloading,
2. vanishing stiffness for $\sigma_1 = \sigma_1^{max}$ (loading) and $\sigma_1 = \sigma_1^{min}$ (unloading)
3. The initial stiffness should have the value E_0.

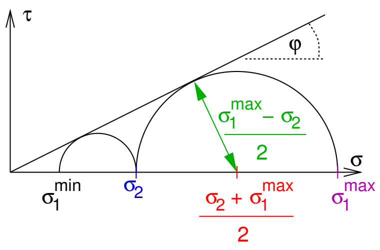

Figure 8: *Mohr-Coulomb* failure criterion for non-cohesive soils

The change of the stiffness is modeled with the trick of using the absolute value function. We specify the vertical stress limits using the *Mohr-Coulomb* failure criterion for non-cohesive soils (fig. 8)

$$\sigma_1^{max} - \sigma_2 = (\sigma_1^{max} + \sigma_2)\sin\varphi \qquad (9)$$

$$\sigma_2 - \sigma_1^{min} = (\sigma_2 + \sigma_1^{min})\sin\varphi \qquad (10)$$

with the friction angle φ. The deviatoric stress $\sigma_1 - \sigma_2$ as well as the sum of the principal stresses $\sigma_1 + \sigma_2$ control the limit state. Thus both terms are used in the formulation:

$$\dot\sigma_1 = a_1(\sigma_1 + \sigma_2)\dot\varepsilon_1 + a_2(\sigma_1 - \sigma_2)|\dot\varepsilon_1| \qquad (11)$$

We determine the coefficients a_1 and a_2 with two conditions: a given initial stiffness and a given limit state.

The initial stiffness is the gradient of the loading branch of the stress-strain curve at $\sigma_1 = \sigma_2$ (fig. 7). Equation (11) reads for loading ($\dot\varepsilon < 0$)

$$\dot\sigma_1 = [a_1(\sigma_1 + \sigma_2) - a_2(\sigma_1 - \sigma_2)]\dot\varepsilon_1 \, .$$

We know from (8) that the term in the square brackets is the stiffness. The initial value at $\sigma_1 = \sigma_2$ should be E_0

$$a_1 2\sigma_2 = E_0 \, .$$

From this follows

$$a_1 = \frac{E_0}{2\sigma_2} \, .$$

In triaxial tests E_0/σ_2 is approximately constant (proportionality of the stiffness to the stress level!).

Equation (11) should also simulate the limit state. This means vanishing stiffness for $\sigma_1 = \sigma_1^{max}$ during loading. With other words, the stress rate $\dot\sigma_1 = 0$ should vanish at maximum stress for negative strain rate $\dot\varepsilon < 0$. With this condition (11) yields

$$a_1(\sigma_1^{max} + \sigma_2) - a_2(\sigma_1^{max} - \sigma_2) = 0 \, ,$$

and with (9)

$$a_2 = \frac{a_1}{\sin\varphi} = \frac{E_0}{2\sigma_2 \sin\varphi} \, .$$

It can be easily checked, that the limit state for unloading is also fulfilled with this coefficient a_2.

The one-dimensional hypoplastic law for the triaxial test reads then:

$$\dot\sigma_1 = \frac{E_0}{2}\frac{\sigma_1 + \sigma_2}{\sigma_2}\dot\varepsilon_1 + \frac{E_0}{2\sin\varphi}\frac{\sigma_1 - \sigma_2}{\sigma_2}|\dot\varepsilon_1| \qquad (12)$$

Limit states can be modeled with the help of deviatoric stress terms.

As an example the stress-strain curve for $E_0 = 1000$ kN/m^2, $\varphi = 30°$ and $\sigma_2 = -100$ kN/m^2 is shown in fig. 9.

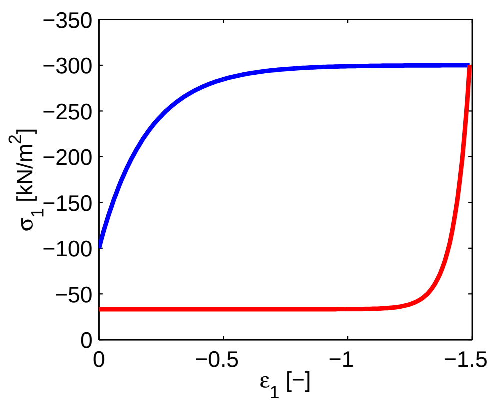

Figure 9: Result of a simulation of the triaxial test with the simple hypoplastic law (11)

4 Summary

The hypoplastic constitutive law formulates the stress-strain behaviour of non-cohesive soils in rate form:

$$\dot{\sigma} = h(\sigma, \dot{\varepsilon})$$

The stress-strain relation follows as a result of a (numerical) integration over the time. Thus it depends on the load history.

From the knowledge of the behaviour of sand in lab tests three main properties follow for the function h:

1. positively homogeneous of the first degree in $\dot{\varepsilon}$ (rate-independence)
2. incremental non-linearly in $\dot{\varepsilon}$ (absolute value function)
3. homogeneous[2] in σ (stiffness depends on stress)

Limit states can be modeled with deviatoric stress terms.

5 Further reading

For scientists, which take risk to become addicted to hypoplasticity, we recommend the comprehensive booklet of *D. Kolymbas*: Introduction to Hypoplasticity [3].

6 Utilities

The homepage of the Institute of Geotechnical and Tunnel Engineering (*http://geotechnik.uibk.ac.at*) offers programs concerning Hypoplasticity:
http://geotechnik.uibk.ac.at/res/hypopl.html

An implementation of the hypoplastic constitutive law [5] for the finite-element program ABAQUS is freely available. This is a FORTRAN subroutine for user-defined constitutive laws. The calibration of the parameters is shown in [2].

[2]Only simple hypoplastic constitutive law versions are homogeneous of the first degree in σ. Newer versions are homogeneous in nth degree in σ. Thus the stress dependence of the stiffness can be better modeled.

Further more two test programs are available, which simulate various element tests (confined compression test, triaxial test, simple shearing) using Hypoplasticity. The program *Hyptest* of Herle for DOS and UNIX/LINUX, as well as the WINDOWS version of Doanh, Herle and Bourgeois.

References

[1] W. Fellin and A. Ostermann. A hypoplasticity routine for ABAQUS with consistent tangent operator and error control, *On-line:* URL http://geotechnik.uibk.ac.at/res/FEhypo.html [2001]

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