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Price-Directed Search and Collusion*

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Abstract

In many (online) markets, consumers can readily observe prices, but need to examine individual products at positive cost in order to assess how well they match their needs. We propose a tractable model of price-directed sequential search in a market where firms compete in prices. Each product meets consumers' basic needs, however they are only fully satisfied with a certain probability. In our setup, four types of pricing equilibria emerge, some of which entail inefficiencies as not all consumers are (always) served. We then lend our model to analyze collusion. We find that for any number of firms, there exists a parameter region in which the payoff-dominant symmetric collusive equilibrium gives rise to a higher expected total social welfare than the repeated one-shot Nash equilibrium. In other regions, welfare is identical under collusion and merely consumer rents are transferred, or both welfare and consumer rents are reduced. An all-inclusive cartel maximizing industry profit increases welfare for an even larger set of parameters, but may also be detrimental to it.

Keywords: Consumer Search, Directed Search, Price Competition, Mixed-Strategy Pricing, Collusion, Cartels

JEL Classification: D43, D83, L13

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1 Introduction

Imagine you are looking for a hotel room to spend a weekend city trip. You open your favorite online platform, specify the city and date, and also select, as far as this is possible on the given platform, all additional criteria that your hotel has to satisfy: at least four stars, inner-city location, free wireless access, etc. You are then presented a list of suitable options, perhaps ordered by price (or at least, with prices prominently displayed), and may click on each individual option to obtain further details (such as pictures and a full description of the hotel's amenities), as well as the ability to book a room.

Of course, it will typically be difficult to assess the attractiveness of any given hotel before inspecting it more closely. For example, only after checking some pictures, a more detailed description and/or customer reviews, you may learn about a hotel's modern style and lively location, which you value (while other consumers may prefer a traditional house in a quiet neighborhood). Yet, your basic needs (e.g., that you can stay at a four-star hotel in the city center) will certainly be satisfied by all showcased alternatives. Moreover, since the presented options appear ex-ante identical (as all satisfy your specified criteria), it will make sense to search through them from lowest to highest price, stopping (booking) when you find something sufficiently nice.[1]

We believe that not only consumers' search for hotel rooms, but many (online) search problems can be described by a similar structure: prices (in a given product category) are readily observable, the competing products all satisfy consumers' basic needs, but consumers need to search through them at positive cost (e.g, as this requires time and effort) to be able to assess how much they like them *beyond* a certain base utility.

In order to capture the spirit of this setting while keeping the analysis tractable, we set up a model of price-directed sequential search in which consumers' *match values* (i.e., how much they like the ex-ante symmetric firms' heterogeneous products) are binary: for each firm's single product, a consumer's valuation can either be low (a partial match) – but

[1] Baye et al. (2009) provide evidence that the number of clicks received by online retailers is indeed highly dependent on their price rank. Examining a large price-comparison site at the time, they find that the lowest-priced retailers for a given product received on average 60% more clicks than higher-priced competitors (even though the authors lack data how this was eventually reflected in final demand). Relatedly, Ellison and Ellison (2009) document that the price transparency provided by a price search engine tended to make demand (for low-quality computer memory modules, a relatively homogeneous good) extremely elastic, even though this was counteracted by obfuscation attempts by some of the examined online retailers.

still, above firms' marginal cost – or high (a full match), following an exogenous two-point distribution. We solve for consumers' optimal search procedure for any combination of prices and exogenous parameters (number of firms, value of partial matches relative to full matches, probability of full matches) and proceed to study firms' equilibrium pricing.

We find that, depending on parameters, one of four types of unique symmetric pricing equilibria emerges. First, if consumers' search costs are large and/or product differentiation is small (in the sense that partial matches provide a similar utility to full matches), firms deterministically price at marginal cost, while consumers search exactly one random firm, buying there no matter whether a full or partial match is found. Marginal-cost pricing occurs because under the described circumstances, no consumer ever searches on after starting at (one of) the lowest-priced firm(s), giving rise to Bertrand-type competition.

If instead the search costs are not too large while product differentiation is sufficiently large, firms can sustain positive profits in equilibrium. Due to undercutting incentives to be searched earlier, firms draw prices randomly from an atomless distribution bounded away from marginal cost, and consumers search orderly from lowest to highest price, only stopping and buying when they find a full match at some firm (but potentially, returning to the lowest-priced firm if no full match is found at any firm). We show that the mixed-strategy equilibrium comes in three subtypes, depending on the degree of product differentiation. For a relatively low differentiation, a "low-price equilibrium" emerges in which all firms always price below the valuation of partially-matched consumers, such that all consumers purchase eventually. In contrast, for a relatively high differentiation, a "high-price equilibrium" results in which all firms always price above the valuation of partially-matched consumers, such that consumers without a full match at any firm drop out deterministically. Finally, for an intermediate differentiation, a "gap equilibrium" occurs in which the firms randomize between pricing below or above the valuation of partially-matched consumers – with a gap just above this valuation – such that consumers without a full match at any firm drop out with positive probability.

In terms of welfare, it is obvious that the latter two equilibria entail inefficiencies, as not all consumers are (always) served, despite all consumers having a valuation above marginal cost (hence, in a second-best scenario in which firms were forced to price below the valuation of partially-matched consumers, welfare would be strictly higher in expectation). The cause of this welfare loss differs from that in existing tractable models of price-directed search such as Ding and Zhang (2018) (see also the detailed literature discussion below), which generally assume an all-or-nothing structure of product matches (either a full match or no match at all).

We also consider the comparative statics of social welfare, firm and industry profits and consumer surplus with respect to the market parameters. For welfare, we find that the comparative statics generally go in the expected direction: welfare increases in the value and probability of full matches, value of partial matches and number of firms, and decreases in consumers' search cost – which we view as strength of our model.[2] As other noteworthy findings, we show that an increase in the number of firms may increase industry profit by expanding the fraction of consumers with at least one full match (and thereby, a high willingness to pay), and that seemingly positive changes for consumers (such as an increase in the value of full matches or a decrease in search costs) may actually harm consumers by dampening competition.

We would like to propose our framework as realistic, yet flexible and tractable model of price-directed search. To showcase its usefulness, we study, as main application, the consequences of collusion and cartelization on market outcomes. Lately, suspicions arose that the increased transparency in online markets has led to firms colluding to keep prices high, such as pointed out by the European Commission[3]:

> "Additionally the price transparency which comes with e-commerce provides possibilities to easily monitor the price setting behaviour of competitors and retailers. Many companies use pricing software which automatically adjusts prices to those of competitors."

Considerable attention has since been given to firms using algorithms which take changes in prices of their competitors into account.[4] The European Commission then conducted inves-

[2] Yet, in the gap-equilibrium region, also counter-intuitive reversed comparative statics may occur locally. This is because firms may strategically respond to changes in some of the parameters by shifting probability mass to high prices (for an increase in the value and probability of full matches) or to low prices (for an increase in search costs), which, by increasing deadweight loss in the former case and decreasing it in the latter, may dominate the positive (negative) direct effects on welfare.

[3] See https://one.oecd.org/document/DAF/COMP/WD(2018)61/en/pdf, p.3., accessed March 14, 2019.

[4] See https://www.fastcompany.com/90311848/yes-retailers-are-colluding-to-inflate-prices-online or https://www.bizcommunity.com/Article/196/394/188110.html. (both accessed March 14, 2019).

tigations on pricing algorithms employed by firms in e-commerce and to which extent they qualify as anti-competitive practices.[5] They found that especially in a horizontal context, the pricing algorithms facilitate explicit *and implicit, tacit* collusive agreements.

While other recent contributions such as Petrikaitė (2016) have also investigated collusion in markets characterized by consumer search, we are, to the best of our knowledge, the first to combine an analysis of price-directed search and collusion. We show that the payoff-dominant symmetric collusive equilibrium (which can e.g. be supported by grim-trigger strategies if firms are sufficiently patient) has firms coordinating on one of three price levels, depending on parameters. For either very low or high product differentiation, the optimal collusive price level coincides with the highest price that still keeps consumers in the market, such that consumer rents are clearly transferred to firms. Moreover, under moderately high differentiation, also social welfare may be reduced, as part of the consumers drop out deterministically in the collusive equilibrium, while they may be served with positive probability in the corresponding (one-shot) Nash equilibrium of the baseline game.

However, interestingly, for intermediate levels of product differentiation, the profit-maximizing collusive price level lies at consumers' valuation for partial matches, such that all consumers are served deterministically under collusion, and welfare is at the second best. When comparing this to the one-shot Nash outcome under the same parameters, it turns out that this type of collusive equilibrium may actually *increase* welfare (and at the very least does not decrease it). Welfare increases when without collusion, the gap equilibrium would be played, in which case firm coordination eliminates the probabilistic deadweight loss which would arise under unconstrained competition. We establish that for any number of firms, there is indeed a parameter region where this occurs, namely when both the value of partial matches relative to full matches is not too high, and the probability of full matches is intermediate.

Finally, we examine the market outcome under an all-inclusive cartel (alternatively, if all firms merge to a multi-product monopolist). Once again, we find that one of three different price configurations is optimal, two of which – again under very low differentiation and high differentiation – have all firms choosing the maximum price which keeps consumers in the market. However, it can no longer be profit-maximizing to collectively price at the valuation

[5] See `https://one.oecd.org/document/DAF/COMP/WD(2017)12/en/pdf`, accessed March 14, 2019.

of partially-matched consumers: the same demand (i.e., all consumers in the market), but a higher profit, can now be achieved when just one firm sets this low price. The low-priced product then serves as "compromise option" for those consumers who don't have a full match at any firm, while all other products are priced maximally and sold to fully-matched consumers. Whenever this is the cartel solution, welfare is at the second best, and we show that a welfare improvement through cartelization occurs for an even larger set of parameters than under symmetric collusion.

Related Literature. Our paper joins an extensive literature on costly consumer search, studying the effects of frictions and incomplete information about product characteristics and/or prices on market outcomes. For comprehensive literature reviews see Anderson and Renault (2018) and Baye et al. (2006), or, for the case of digital markets, Moraga-González (2018).

In early work which relates to our model, such as the seminal papers by Wolinsky (1986), Stahl (1989) and Anderson and Renault (1999), prices are unobservable and consumer search is random. Departing from models of random search, there have been efforts to describe environments in which consumers search firms according to some order. The first papers in this vein focused on predetermined orders, arising naturally e.g. when thinking about geographical distance (see Arbatskaya (2007) for homogeneous products, Armstrong et al. (2009) for differentiated products with a "prominent" firm[6], or Zhou (2011) for a general analysis with differentiated products). In Athey and Ellison (2011) and Chen and He (2011), firms bid for positions along consumers' search path. However, in these models, prices do not influence the order of search. Armstrong (2017) outlines a setting in which the order of search is chosen endogenously by consumers forming *expectations* about prices and firms acting according to their beliefs in equilibrium.

One of the first attempts to model *observable* prices as important strategic variables for directing search can be found in Armstrong and Zhou (2011, Section 2), where firms advertise the price of their differentiated product on a price-comparison website. Consumers' optimal search path is then guided by those advertised prices. To keep the model tractable, Armstrong and Zhou introduce a specific (Hotelling duopoly) structure in which consumers'

[6]That is, one firm is exogenously searched first by all consumers, while the remaining firms are searched in random order.

match values are perfectly negatively correlated.[7] A main finding is that the competition among firms to receive a larger market share by being sampled first drives down retail prices, relative to a benchmark model without price advertising, and that this effect is stronger when search frictions increase.

Tractability is generally a major issue when it comes to solving models of price-directed search. For example, even a duopoly version of the standard differentiated-products framework by Wolinsky (1986) with independently distributed match values becomes intractable with observable prices, as the resulting mixed-strategy equilibrium is extremely hard to characterize. Haan et al. (2018) and Choi et al. (2018) circumvent this problem by incorporating sufficiently strong ex-ante differentiation into Wolinsky's framework with observable prices.[8] This restores existence of a pure-strategy equilibrium that can be characterized. By considering a two-point distribution of match values, we obtain tractability without introducing any exogenous ex-ante differentiation.

In terms of its underlying model, our paper is most closely related to Ding and Zhang (2018), which also studies price-directed search in a market with differentiated products and ex-ante homogeneous firms. There are two major differences. First, we do not include informed consumers who costlessly observe all match values, which is however crucial to generate most interesting results in Ding and Zhang.[9] Second, and most important, we allow for product differentiation to be more nuanced. While in Ding and Zhang consumers either fully value a product or not at all, in our setting they may have a positive willingness to pay for all products. As a result, consumers may optimally *return* to purchase from a previously sampled firm, which affects competition and has important consequences for market outcomes that are otherwise not captured.[10]

[7] More concretely, upon inspecting the lower-priced product first, consumers learn its match value and can then perfectly deduce the match value offered by the other firm.

[8] See also Shen (2015) for related analysis in a Hotelling context.

[9] For example, they also find circumstances where a "gap equilibrium" occurs, but this is only the case if the fraction of informed consumers is sufficiently large. Especially for markets where many consumers are first-time buyers, such as our motivating example of lodging services, consumers who know their match values in advance (or can search them for free) arguably constitute a small minority. For simplicity and to highlight a different channel, we set their number to zero in our model. For a sufficiently small fraction of informed consumers, our qualitative results would remain similar.

[10] As a third distinction to Ding and Zhang (2018), consumers' first search is not costless in our model. While we anyway consider a positive search cost for every sampled product to be more realistic, the equilibrium characterization of our baseline model would remain virtually unchanged with costless first search (just the parameter region where the market is inactive would vanish). Of course, our comparative-statics analysis of various welfare measures, as well as the specification of the optimal collusive schemes, would have to be adapted.

Our analysis of firm coordination is related to a vast theoretical literature on collusion.[11] However, applications to markets with search frictions have been limited, and mainly focused on the *sustainability* of collusion rather than on its welfare effects. Petrikaitė (2016) contrasts models of non-directed consumer search with differentiated and homogeneous products to investigate how cartel stability is affected by search costs. She finds that increased search costs facilitate collusion if products are differentiated, while the opposite is true if products are homogeneous. In the homogeneous-products duopoly model with imperfect monitoring studied by Campbell et al. (2005), increased search costs also make collusion harder to sustain. Schultz (2005) considers a Hotelling framework in which only a fraction of consumers is aware of both firms' prices.[12] He finds that an increase in market transparency through a higher fraction of informed consumers decreases the scope for collusion. In a model of non-sequential search for homogeneous products, Nilsson (1999) finds the opposite, namely that an increase in market transparency through lower search costs may promote collusion. Overall, the specific market environment seems decisive.[13]

In contrast to the aforementioned papers, our model features observable prices *for all* market participants. It is thus easy for firms to detect and punish deviations from the (tacit or explicit) collusive agreements. Further, prices actually *direct* search. Compared to models of random search, this has a notable effect on firms' incentives to collude, as they are able to directly influence consumers' search order by deviating from a collusive agreement. A major novel finding in our paper is that firms avoiding competition can have no *or even a positive* effect on total welfare.

The remainder of this article is structured as follows. Section 2 introduces the model setup, while in Section 3, we solve the baseline model and discuss its welfare implications and comparative statics. In Section 4, we reformulate our baseline model as infinitely repeated game and analyze the payoff-dominant symmetric equilibria, as well as the cartel outcome. We also compare welfare to the baseline model. Section 5 concludes. Several technical proofs are relegated to the Appendix.

[11] See Feuerstein (2005) for a survey.

[12] See also Schultz (2017), which extends the author's earlier model to allow for limited price observability on the supply side as well.

[13] The question whether or not collusion is facilitated in markets with costly consumer search has also been addressed empirically. Moraga-González et al. (2015) investigate the market for car dealerships by employing models of random search. While they do not find strong evidence for collusion in the examined markets, they show that lowering consumers' search costs can indeed lead to higher prices, as is consistent with our baseline results. Nishida and Remer (2018) also find detrimental effects of reduced search costs studying the retail gasoline market.

2 Model Setup

Consider the following market. There are $n \geq 2$ risk-neutral firms $i = 1, \ldots, n$ that compete in prices p_i. Each firm offers a single differentiated product. Firms' constant marginal costs of production are normalized to 0.

There is a unit mass of risk-neutral consumers with unit demand and an outside-option value that is normalized to zero. Each consumer freely observes the prices of all products. However, consumers are initially unaware whether any given product will be a full or partial match for them. Precisely, product i perfectly suits a consumer's needs (the product is "a full match") with probability $\alpha \in (0,1)$. In case of a full match, consumers' willingness to pay is given by $v_i = v_H > 0$. With complementary probability $1-\alpha$, product i is only "a partial match", for which consumers' willingness to pay is given by $v_i = v_L \in [0, v_H]$. We assume that the match values v_i are identically and independently distributed across each consumer-firm pair, and that the firms are unable to identify which product(s) will be a match for any individual consumer, ruling out price discrimination.

In order to find out their match values, consumers have to incur a search cost $s \geq 0$ per product that they sample. It is assumed that they cannot purchase any product before searching it first. Consumers engage in optimal sequential search with free recall and maximize their expected consumption utility, where consumption utility is given by

$$u_i := v_i - p_i - ks, \qquad \text{with } v_i \in \{v_L, v_H\} \tag{1}$$

when buying product i (which can either be a full or partial match) after having searched $k \in \{1, \ldots, n\}$ products, and $u_0 = -ks$ when taking their outside option after having searched $k \in \{0, \ldots, n\}$ products. All market parameters are common knowledge.

The timing of the game is as follows. First, firms simultaneously set prices p_i. Second, consumers observe these prices, and engage in optimal sequential search. Third, payoffs realize.

In order to make the problem interesting, we finally assume that $\alpha v_H + (1-\alpha)v_L - s \geq 0$. Otherwise, the market collapses, as no firm could offer a non-negative expected surplus to consumers even when setting $p_i = 0$.

3 Equilibrium Analysis

Optimal Search. Since, apart from their prices, firms' products appear ex-ante identical, consumers will clearly find it optimal to search firms in ascending order of their prices.[14] Without loss of generality, we index firms such that $p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq p_n$. Given a consumer started at firm 1 and found a full match, the consumer optimally purchases, since there can be no gain from searching on. However, if only a partial match is found at firm 1, the consumer might want to continue to search firm 2, and so on. Consumers' optimal search behavior now crucially depends on whether $p_1 > v_L$ or $p_1 \leq v_L$, as only in the latter case, consumers may want to return to purchase at firm 1 in the course of their search process. The following lemma fully characterizes consumers' optimal search behavior.

Lemma 1. *Optimal Search:*

- *If $p_1 > v_L$, search, in increasing order of prices, all firms $i = 1, \ldots, n$ for which $p_i \leq v_H - \frac{s}{\alpha}$. Purchase immediately if a full match is found, and search on if not. If no full match is found at any suitable firm, take the outside option.*
- *If $p_1 \leq v_L$, start search at firm 1 if $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$, and otherwise take the outside option. Given firm 1 is searched and a full match is found, purchase there immediately. If not, search, in increasing order of prices, all firms $i = 2, \ldots, n$ for which $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$. Purchase immediately if a full match is found, and search on if not. If no full match is found at any suitable firm, purchase at firm 1.*

Proof. The first part is straightforward: Given that all prices exceed v_L, consumers will only buy from a firm if it provides a full match, and as long as no full match has been found, consumers hold a utility of zero. Hence, provided that no full match has been found yet, the expected one-shot gains from searching any firm i are given by $\alpha(v_H - p_i) - s$, which is non-negative if and only if $p_i \leq v_H - \frac{s}{\alpha}$. It is therefore optimal to search, in increasing order of prices, all firms for which this holds, and purchase immediately if a full match is found. If no full match is found at any firm which satisfies $p_i \leq v_H - \frac{s}{\alpha}$, a consumer optimally takes the outside option.

[14] In case of ties, consumers are assumed to randomize with equal probability between firms, which is however inconsequential for our results. We moreover assume that whenever a consumer is indifferent between purchasing directly and searching on, the consumer searches on, and whenever a consumer is indifferent between buying and not buying after their search process, the consumer buys.

If, on the other hand, $p_1 \leq v_L$, the expected one-shot gains of searching any firm are clearly largest for firm 1 and if no other firm has been searched yet. Hence, a consumer should only start to search (at firm 1) if the expected one-shot gains of doing so, $\alpha(v_H - p_1) + (1-\alpha)(v_L - p_1) - s$, are non-negative. This transforms to $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$. If this holds and it is therefore optimal to search firm 1, consumers should clearly purchase there immediately if a full match is found. If a partial match is found, a consumer holds a purchase option of value $v_L - p_1 \geq 0$, which remains true as long as only partial matches have been found at every searched firm. Hence, provided that only partial matches have been found so far, the expected one-shot gains from searching any firm $i = 2, \dots, n$ are given by $\alpha((v_H - p_i) - (v_L - p_1)) - s$. This is non-negative for all firms i which satisfy $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$. It is therefore optimal to search these firms in increasing order of their prices and purchase immediately if a full match is found. If no full match is found at any firm which satisfies $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$, a consumer optimally returns to purchase from firm 1. □

Preliminary Equilibrium Results. Having characterized consumers' optimal search behavior, one may first note that for $v_H - v_L - \frac{s}{\alpha} \leq 0$, the binding condition for consumers to start searching is $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$ $(\leq v_L)$; moreover, consumers will never search firms that are not among the lowest-priced. The reason is that in the considered parameter range, after obtaining a partial match at (one of) the lowest-priced firm(s), the expected gains from searching are too low for any higher-priced firms. Intuitively, this is true because the condition $v_H - v_L - \frac{s}{\alpha} \leq 0$ holds if either the probability of finding a full match is very low relative to the search cost ($\alpha \leq \frac{s}{v_H}$), or if this not the case, but partial matches provide a too similar utility to full matches, given the probability of finding a full match and the search cost ($\alpha > \frac{s}{v_H}$, but $\frac{v_L}{v_H} \geq 1 - \frac{s}{\alpha v_H}$). Then, the property that consumers will only search firms which are among the lowest-priced immediately implies the following.

Proposition 1. *Suppose that $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \overline{\gamma} := 1 - \frac{s}{\alpha v_H}$. Then in the unique symmetric equilibrium each firm chooses $p^* = 0$ and earns zero profit. On the equilibrium path, each consumer searches exactly one random firm and buys there immediately, independent of whether a full or partial match is found.*[15]

[15] In the borderline case where $\frac{v_L}{v_H} = \overline{\gamma}$, given that $p_i = 0$ for all firms, consumers are actually indifferent between buying immediately after obtaining a partial match or searching on. This is however inconsequential.

Proof. See the argument above. Given $p^* = 0$, consumers indeed find it optimal to search one random firm due to the parameter assumption of $\alpha v_H + (1-\alpha)v_L - s \geq 0$. $\square$

We will subsequently refer to the parameter region where Proposition 1 holds as "Bertrand region", since intense price competition drives firms to price at marginal cost. As we show next, the market outcome is decisively different for all other parameter combinations.

Lemma 2. *If $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \overline{\gamma}$, there exists no symmetric pure-strategy equilibrium. In a symmetric mixed-strategy equilibrium, firms make positive expected profits and draw prices from an atomless CDF bounded away from zero.*

Proof. A symmetric pure strategy-equilibrium at any positive price level can never exist because either firm would have an incentive to marginally undercut to be searched first by all consumers, rather than just by $1/n$ of the consumers. However, unlike the case where $v_H - v_L - \frac{s}{\alpha} \leq 0$, it is also no equilibrium that every firm prices at marginal cost (i.e., zero). This is because, for $v_H - v_L - \frac{s}{\alpha} > 0$, when all rival firms price at zero, setting a price in the non-empty range $(0, v_H - v_L - \frac{s}{\alpha}]$ guarantees a firm to be searched (by those consumers who did not find a full match at any rival firm, compare with Lemma 1) and make a positive profit. Hence, any symmetric equilibrium must be in mixed strategies. The respective equilibrium pricing CDF must be bounded away from zero because firms can guarantee a positive profit. It must be atomless because otherwise, transferring probability mass from the atom(s) to prices marginally below would pay because this avoids ties. $\square$

Preview of Mixed-Strategy Equilibria. It turns out that the symmetric mixed-strategy equilibrium for the case that $v_H - v_L - \frac{s}{\alpha} > 0$ comes in three qualitatively different subtypes, depending on the degree of product differentiation (which is inversely related to v_L/v_H) in combination with the other market parameters.

In particular, as mentioned in the Introduction, either a "high-price equilibrium" (high differentiation, with $\frac{v_L}{v_H} \leq \underline{\gamma}$), a "low-price equilibrium" (relatively low differentiation, with $\frac{v_L}{v_H} \in [\tilde{\gamma}, \overline{\gamma})$), or a "gap equilibrium" (intermediate differentiation, with $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$) emerges as the unique equilibrium. In the next three subsections, we fully characterize these equilibria in turn. Figure 1 previews the various equilibrium regions in $(\alpha, \frac{v_L}{v_H})$-space for a specific combination of search costs (relative to v_H) and number of firms. Note that in region X in the bottom-left corner, our parameter assumption of $\alpha v_H + (1-\alpha)v_L - s \geq 0$ is violated, such that the market is inactive in this region.

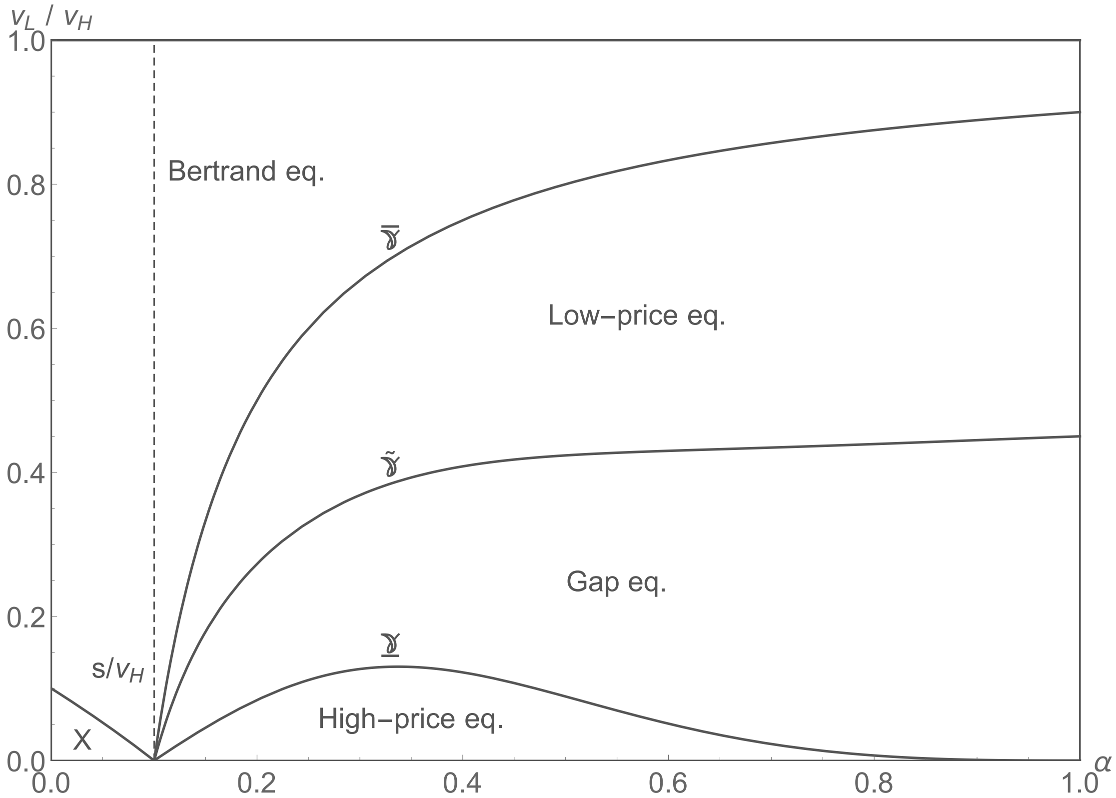

Figure 1: Depiction of equilibrium regions for $\frac{s}{v_H} = 0.1$ and $n = 4$.

3.1 High-Price Equilibrium

As we show, if product differentiation is relatively high, a "high-price equilibrium" emerges. In such an equilibrium, all firms always price strictly above v_L, such that only consumers with a full match may ever buy at any given firm. Moreover, a firm cannot attract any "returning" demand: consumers either buy immediately after having searched some firm, which we refer to as "fresh" demand, or never return. We now construct such a candidate equilibrium.

Since a symmetric equilibrium price distribution must be atomless (see Lemma 2), a firm pricing at the respective upper support bound $\overline{p}_H$ will only be searched by consumers who did not have a full match at any (earlier sampled) rival firm. But then it follows immediately that $\overline{p}_H$ must be equal to the highest price that may ever be searched by consumers, namely

$\overline{p}_H = v_H - \frac{s}{\alpha}$ (compare with Lemma 1). This is because if $\overline{p}_H$ was smaller, a firm choosing $\overline{p}_H$ could profitably deviate upward to the price $v_H - \frac{s}{\alpha}$ instead, for which it would not lose any demand. On the other hand, if $\overline{p}_H$ was larger, a firm setting this price would never be searched.

Having pinned down the equilibrium upper support bound, firms' equilibrium expected profit easily follows: it is given by the highest equilibrium price times the number of consumers a firm can serve at it (the mass of consumers who have no full match at firms 1 to $n-1$, but a full match at firm n), such that $\pi_H^* = \overline{p}_H(1-\alpha)^{n-1}\alpha$. In turn, also firms' equilibrium lower support bound can easily be derived: when a firm chooses the lowest equilibrium price $\underline{p}_H (> v_L)$, it will be searched first by all consumers, and a fraction α of those (with a full match) will purchase there. Hence, $\underline{p}_H$ simply solves $\underline{p}_H\alpha = \pi_H^*$.

The equilibrium CDF $F_H(p)$ can then be found as follows. A firm pricing at some price p in the support of F_H gets an expected profit of

$$\pi_i(p) = p(1-\alpha F_H(p))^{n-1}\alpha,$$

which has to be equal to π_H^* everywhere in the support – solving $\pi_i(p) = \pi_H^*$ for $F_H(p)$ then gives the equilibrium CDF. To understand the above profit expression, consider firm i's probability to sell to any given consumer. Clearly, a consumer will only search firm i if there is not a single rival firm with a lower price that also provides a full match to the consumer. The probability that a single rival firm does not have a lower price *and* provides a full match is given by $1-\alpha F_H(p)$. The probability that none of the $n-1$ rival firms does so is then given by $(1-\alpha F_H(p))^{n-1}$. Hence, with the latter probability, any given consumer searches firm i. This consumer will then buy at firm i if it provides a full match to the consumer, for a total purchase probability of $(1-\alpha F_H(p))^{n-1}\alpha$.

Finally, to see when this equilibrium exists, note that as long as it is well-defined such that $\underline{p}_H > v_L$, pricing in the range $(v_L, \underline{p}_H)$ cannot be optimal, as this does not increase a firm's expected demand. However, it can potentially be optimal to price at v_L or below. Pricing at v_L or below has the advantage that a firm also attracts "returning" demand: those consumers who do not have a full match at the deviating firm will still return if they also have no full match at any other (higher-priced) rival firm. While it may seem intuitive that deviating to a strictly lower price than v_L cannot be better than pricing at v_L, this is actually not immediately obvious, as under the high-price candidate equilibrium, deviation prices strictly below v_L imply a positive probability that consumers will return to purchase from the deviating firm (providing a partial match) without searching all rival firms. In the proof of the subsequent proposition, we show however that the optimal deviation price below $\underline{p}_H$ is indeed always given by v_L, for a maximal deviation profit of $\pi_i^{dev^*} = v_L[\alpha + (1-\alpha)^n]$. This is not higher than the candidate equilibrium profit if and only if $\frac{v_L}{v_H} \leq \underline{\gamma}$, where $\underline{\gamma}$ is defined below. Proposition 2 summarizes the above findings.

Proposition 2. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} \in [0, \underline{\gamma}]$, *where*

$$\underline{\gamma} := \frac{\left(1 - \frac{s}{\alpha v_H}\right)(1-\alpha)^{n-1}\alpha}{(1-\alpha)^n + \alpha}. \tag{2}$$

Then there exists a unique symmetric equilibrium in which each firm samples prices continuously from the interval $[\underline{p}_H, \overline{p}_H]$ *following the atomless CDF*

$$F_H(p) := \frac{1}{\alpha}\left[1 - (1-\alpha)\left(\frac{v_H - \frac{s}{\alpha}}{p}\right)^{\frac{1}{n-1}}\right], \tag{3}$$

with

$$\underline{p}_H := \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1} > v_L \tag{4}$$

and

$$\overline{p}_H := v_H - \frac{s}{\alpha}. \tag{5}$$

Each firm makes an expected profit of

$$\pi_H^* := \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1}\alpha. \tag{6}$$

On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and takes the outside option if no full match is found at any firm.

Proof. See Appendix A. □

Note that various versions of the above pricing equilibrium have appeared before in the literature, where it was generally assumed that $v_L = 0$. Setting $v_L = 0$, the equilibrium in Proposition 2 applies whenever $\alpha > \frac{s}{v_H}$. It is then easy to see that we nest the model of price-directed search by Ding and Zhang (2018) for the case in which there are no informed consumers ($\mu = 0$ in their notation).[16] Letting moreover $s = 0$ and $n = 2$, we nest a duopoly version of Varian (1980) with inelastic demand up to a maximum valuation of v_H (with a fraction $\lambda = \frac{\alpha^2}{1-(1-\alpha)^2} = \frac{\alpha}{2-\alpha}$ of fully-informed "shoppers"). For $s = 0$ and arbitrary $n \geq 2$, our setup is also identical to the second stage of Ireland (1993) when his "information shares" s_i (i.e., the share of consumers who know about the existence of firm i) satisfy $s_i = \alpha$ for all $i = 1, ..., n$ (and $v_H = 1$ to match his normalization).[17]

3.2 Low-Price Equilibrium

As we show next, when product differentiation is relatively low, a "low-price equilibrium" results. In this equilibrium, all firms always sample prices below v_L, such that all consumers buy at some firm. It turns out that the equilibrium can then be pinned down by two conditions. The first is that, perhaps surprisingly, the equilibrium lower pricing support bound $\underline{p}_L$ and the equilibrium upper pricing support bound $\overline{p}_L$ lie just so far apart that when having a partial match at the lowest possible price $\underline{p}_L$, a consumer would exactly be indifferent between purchasing there, or searching a firm with the highest possible price $\overline{p}_L$. This implies that on the equilibrium path each consumer keeps searching deterministically until a full match is found, and only returns to the lowest-priced firm in case no full match is found at any firm. Comparing with consumers' optimal search rule for the case where $p_1 \leq v_L$ (see Lemma 1), the relevant condition for this is that

$$\alpha((v_H - \overline{p}_L) - (v_L - \underline{p}_L)) = s. \tag{7}$$

[16] To be precise, consider Ding and Zhang (2018, Proposition 2) for $\mu = 0$, and let $V = v_H$, $\theta = \alpha$ and $N = n$ to match our notation. Then $r = v_H - \frac{s}{\alpha}$ (compare with their equation (2)), and their threshold value s_1' equals αv_H such that part (i) of their Proposition 2 applies. It is then immediate that their equilibrium CDF $R(p)$ coincides with our equilibrium CDF $F_H(p)$ in the high-price equilibrium (and of course, also the equilibrium expected profits are identical).

[17] For $n = 2$ and $s_1 = s_2 = \alpha$, it is straightforward to see that Ireland's second-stage solution coincides with ours (compare with (Ireland, 1993, p.66)). For $n > 2$, this should also be the case, but due to his focus on asymmetric information shares, the comparison of equilibria is less obvious.

Building on this, the second condition is straightforward: A firm's expected profit when pricing at $\underline{p}_L$ is then given by $\pi_i(\underline{p}_L) = \underline{p}_L(\alpha + (1-\alpha)^n)$, as it will have the lowest price, and therefore be sampled first by all consumers with certainty (for a "fresh" demand of α and a "returning" demand of $(1-\alpha)^n$). This must be equal to the firm's expected profit when pricing at $\overline{p}_L$, which is then, since the firm will have the highest price deterministically, given by $\pi_i(\overline{p}_L) = \overline{p}_L(1-\alpha)^{n-1}\alpha$. Hence, it is required that

$$\underline{p}_L(\alpha + (1-\alpha)^n) = \overline{p}_L(1-\alpha)^{n-1}\alpha. \tag{8}$$

Simultaneously solving equations (7) and (8) gives the candidate $\underline{p}_L$, $\overline{p}_L$ and equilibrium profit π_L^*. The candidate equilibrium is well-behaved (in the sense that $\underline{p}_L \geq 0$ and $\overline{p}_L \leq v_L$) under the condition on $\frac{v_L}{v_H}$ that will be provided in the subsequent proposition. In the proof of the proposition, we also show that deviation prices outside the equilibrium support are not profitable: while pricing below $\underline{p}_L$ still increases a firm's expected demand as this leads to a positive probability that consumers with only a partial match at that firm will return before sampling all rival firms, the respective loss of margin more than outweighs the positive effect on demand. Similarly, while pricing above $\overline{p}_L$ still generates a positive expected demand, this demand decreases sufficiently fast such as to render such deviations unprofitable. This is because by pricing above $\overline{p}_L$, a firm risks that even consumers who did not have a full match at any rival firm will not search it, as they may rather prefer to return to the lowest-priced rival (with only a partial match).

Unfortunately, for the general n-firm case, the equilibrium CDF $F_L(p)$ cannot be obtained in closed form.[18] On the other hand, it is implicitly defined by a straightforward condition, and it is easy to see that it is well-behaved (strictly increasing) for any number of firms. In particular, note that when a firm chooses any price $p \leq v_L$ in the support of F_L, its expected profit is given by

$$\pi_i(p) = p\left[\alpha(1-\alpha F_L(p))^{n-1} + (1-F_L(p))^{n-1}(1-\alpha)^n\right],$$

which, for the equilibrium CDF $F_L(p)$, needs to be equal to π_L^*. To understand the above profit equation, notice that the first term of expected demand at price p in the squared bracket

[18]For $n = 2$ ($n = 3$), it can however be obtained as the solution to a simple linear (quadratic) equation.

follows the logic outlined in the description of the high-price equilibrium. This corresponds to a firm's "fresh" demand: those consumers who search firm i (because they have no full match at any lower-priced rival firm, if any) and have a full match at firm i. The second term of expected demand in the squared bracket is "returning" demand. Indeed, with a probability of $(1-F_L(p))^{n-1}$, all rival firms choose a higher price than p. Then, the considered firm i will attract the mass $(1-\alpha)^n$ of consumers who did not find a full match at any firm (including firm i), who ultimately return to firm i.[19] Proposition 3 summarizes our findings.

Proposition 3. *Suppose that $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in [\tilde{\gamma}, \bar{\gamma})$, where*

$$\tilde{\gamma} := \frac{\left(1-\frac{s}{\alpha v_H}\right)\left[\alpha + (1-\alpha)^n\right]}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha} \tag{9}$$

$$\bar{\gamma} := 1 - \frac{s}{\alpha v_H}. \tag{10}$$

Then there exists a unique symmetric equilibrium in which each firm samples prices continuously from the interval $[\underline{p}_L, \overline{p}_L]$ following an atomless CDF $F_L(p)$ that is defined implicitly by

$$p\left[\alpha(1-\alpha F_L(p))^{n-1} + (1-F_L(p))^{n-1}(1-\alpha)^n\right] = \pi_L^*, \tag{11}$$

where

$$\pi_L^* := \frac{(1-\alpha)^{n-1}\left[(1-\alpha)^n+\alpha\right]\left[\alpha(v_H-v_L)-s\right]}{(1-\alpha)^{n-1}(1-2\alpha)+\alpha} \tag{12}$$

denotes each firm's equilibrium expected profit,

$$\underline{p}_L := \frac{(1-\alpha)^{n-1}\left[\alpha(v_H-v_L)-s\right]}{(1-\alpha)^{n-1}(1-2\alpha)+\alpha} \tag{13}$$

and

$$\overline{p}_L := \frac{\left[(1-\alpha)^n+\alpha\right]\left[\alpha(v_H-v_L)-s\right]}{\alpha\left[(1-\alpha)^{n-1}(1-2\alpha)+\alpha\right]} \leq v_L. \tag{14}$$

[19] Note that, since the probability that exactly $k \in \{0,\dots,n-1\}$ of the $n-1$ rival firms price lower than p is given by $\binom{n-1}{k} F_L(p)^k (1-F_L(p))^{n-1-k}$, the "fresh" demand can alternatively be found by computing

$$\sum_{k=0}^{n-1} \binom{n-1}{k} F_L(p)^k (1-F_L(p))^{n-1-k}(1-\alpha)^k\alpha = \alpha\left[(1-\alpha)F_L(p) + (1-F_L(p))\right]^{n-1} = \alpha(1-\alpha F_L(p))^{n-1},$$

where the first equality follows from the binomial theorem. In the event that $k=0$, which happens with probability $(1-F_L(p))^{n-1}$, next to the corresponding "fresh" demand of α, a firm also receives the "returning" demand $(1-\alpha)^n$.

On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and returns to purchase at the lowest-priced firm if no full match is found at any firm.

Proof. See Appendix A. □

3.3 Gap Equilibrium

The arguably most interesting equilibrium arises if v_L takes on intermediate values ($\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$). Then, the high-price equilibrium as characterized in Proposition 2 does not exist because a deviation to v_L is profitable (since $\frac{v_L}{v_H} > \underline{\gamma}$), while the low-price equilibrium as characterized in Proposition 3 does not exist because it would hold that $\overline{p}_L > v_L$ (since $\frac{v_L}{v_H} < \tilde{\gamma}$).

Instead, in the resulting "gap equilibrium" firms draw prices from two disconnected intervals: they either choose low prices in some range $[\underline{p}_M, v_L]$ up to v_L, or they choose high prices in some range $[\underline{p}'_M, \overline{p}_M]$ strictly above v_L. Intuitively, firms' pricing support has a gap right above v_L because firms' demand drops discretely at v_L. This is because when pricing at v_L, there is some positive probability that a firm has the lowest price *and* attracts the mass $(1-\alpha)^n$ of "returning" demand with no full match at any firm, while when pricing at $v_L + \varepsilon$, no consumers would return even if the firm had the lowest price in the market.

Also the gap equilibrium satisfies the two conditions that hold for the low-price equilibrium: the range of firm's equilibrium pricing, $\overline{p}_M - \underline{p}_M$, is again exactly[20] such that on the equilibrium path, every consumer keeps searching until a full match is found (compare with equation (7)), while naturally, pricing at the lowest equilibrium price $\underline{p}_M$ must yield the same expected profit as pricing at the highest equilibrium price $\overline{p}_M$ (compare with equation (8)). It therefore turns out that the equilibrium lower support bound, upper support bound and profit take on the same functional form as in the low-price equilibrium (but now with $\overline{p}_M = \overline{p}_L > v_L$). At the bottom of the upper pricing interval $\underline{p}'_M \in (v_L, \overline{p}_M)$, the expected profit (without any "returning" demand, but with equal "fresh" demand as when pricing at v_L) must then be equal to the expected profit when pricing at v_L (with "returning" demand).

[20] With this, we mean that like in the low-price equilibrium, $\underline{p}_M$ and $\overline{p}_M$ lie just so far apart that when having a partial match at the lowest possible price $\underline{p}_M$, a consumer would exactly be indifferent between purchasing there, or searching a firm with the highest possible price $\overline{p}_M$.

The equilibrium CDFs for the two intervals are finally obtained by essentially the same profit-indifference conditions as in the low-price equilibrium (lower interval) and high-price equilibrium (upper interval), respectively. The only difference is that in the upper interval, the expected profit at any equilibrium price needs to be equal to the equilibrium profit $\pi_M^* = \pi_L^*$, rather than π_H^* in the high-price equilibrium. We further show that in the relevant parameter range, the gap equilibrium is well-behaved, and that there are no profitable deviation prices outside the equilibrium support. Proposition 4 summarizes our findings. A graphical illustration of the equilibrium CDF in the gap equilibrium is provided in Figure 2.

Proposition 4. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$. *Then there exists a unique symmetric equilibrium in which each firm samples prices from two disconnected intervals* $[\underline{p}_M, v_L] \cup [\underline{p}'_M, \overline{p}_M]$, *with* $\underline{p}'_M > v_L$. *In the lower interval, firms draw prices from the atomless CDF* $F_{M_1}(p) := F_L(p)$ *as defined in Proposition 3, while in the upper interval, firms draw prices from the atomless CDF*

$$F_{M_2}(p) := \frac{1}{\alpha}\left[1 - \left(\frac{\pi_L^*}{\alpha p}\right)^{\frac{1}{n-1}}\right], \tag{15}$$

with π_L^*, $\underline{p}_M := \underline{p}_L$ *and* $\overline{p}_M := \overline{p}_L$ *as defined in Proposition 3 and*

$$\underline{p}'_M := \frac{\pi_L^*}{\alpha(1-\alpha\kappa)^{n-1}}, \tag{16}$$

where $\kappa := F_{M_1}(v_L)$ *is implicitly defined by*

$$v_L\big[\alpha(1-\alpha\kappa)^{n-1} + (1-\kappa)^{n-1}(1-\alpha)^n\big] - \pi_L^* = 0, \tag{17}$$

and $F_{M_2}(\underline{p}'_M) = F_{M_1}(v_L) = \kappa$. *Each firm makes an expected profit of* $\pi_M^* := \pi_L^*$. *On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and returns to purchase at the lowest-priced firm if* $p_1 \le v_L$ *and no full match is found at any firm.*

Proof. See Appendix A. □

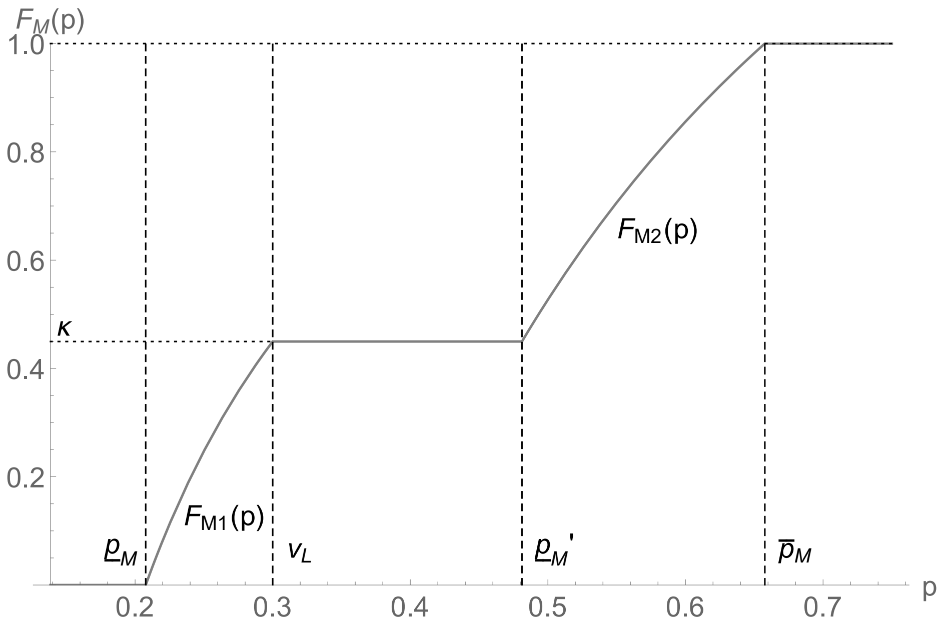

Figure 2: Example equilibrium CDF in the gap equilibrium. The parameters used are $v_H = 1$, $v_L = 0.3$, $s = 0.1$, $\alpha = 0.4$, $n = 2$.

3.4 Welfare and Comparative Statics

In this subsection, we employ our equilibrium characterization to discuss total social welfare, firm profits/producer surplus as well as consumer surplus and how they depend on the model parameters v_H, v_L, s, α and n. Fortunately, the equilibrium total social welfare is easily obtained: since all prices paid are pure transfers, it is given by the aggregate match values realized through consumption minus the total search costs incurred.

In the "Bertrand region" where Proposition 1 applies, each consumer searches only one random firm, obtains a match value of v_H or v_L with probability α and $1-\alpha$, respectively, and buys there deterministically. Hence, total social welfare in the Bertrand region equals $W_B = \alpha v_H + (1-\alpha)v_L - s$. In all other regions, we have established that each consumer keeps searching until a full match is obtained (if at any firm). In these regions, the aggregate search friction incurred is thus given by[21]

$$S = \left(\sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1} ks\right) + (1-\alpha)^{n-1} ns = s\left[\frac{1-(1-\alpha)^n}{\alpha}\right], \tag{18}$$

[21]Note that for $k = 1, \ldots, n-1$, a fraction $(1-\alpha)^{k-1}\alpha$ of consumers has no full match at the first $k-1$ sampled firms and a full match at the k'th sampled firm, with a per-consumer search cost of ks (first term). A fraction $(1-\alpha)^{n-1}$ of consumers has no full match at the first $n-1$ firms and therefore searches all firms, with a per-consumer search cost of ns (second term).

where the second equality can easily be shown via induction starting from $n = 2$.

At the same time, the realized aggregate match values depend on the equilibrium which is played. In the high-price equilibrium, a fraction $(1-\alpha)^n$ of consumers does not find a full match at any firm and therefore drops out of the market, such that the aggregate match values realized are given by $v_H\left[1-(1-\alpha)^n\right]$. In the low-price equilibrium, once again a fraction $(1-\alpha)^n$ of consumers does not find a full match at any firm, but now these consumers will also buy with their partial match (at the lowest-priced firm). Hence, the aggregate match values realized are given by $v_H\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n$. Finally, in the gap equilibrium, the fraction $(1-\alpha)^n$ of consumers who do not have a full match at any firm will only buy with their partial match if the lowest-priced firm prices below v_L, which happens with probability $1-(1-\kappa)^n$. Hence, the *expected* aggregate match values realized in this case are given by $v_H\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n\left[1-(1-\kappa)^n\right]$. Subtracting the aggregate search friction S from these aggregate match values, the subsequent lemma is immediate.

Lemma 3. *Total social welfare in the market is given by*

$$W = \begin{cases} \alpha v_H + (1-\alpha)v_L - s & \text{if } \alpha \le \frac{s}{v_H}, \text{ or } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \ge \bar{\gamma} \\ \left(v_H - \frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right] & \text{if } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \le \underline{\gamma} \\ \left(v_H - \frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right] + v_L(1-\alpha)^n\left[1-(1-\kappa)^n\right] & \text{if } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma}) \\ \left(v_H - \frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right] + v_L(1-\alpha)^n & \text{if } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \in [\tilde{\gamma}, \bar{\gamma}). \end{cases} \tag{19}$$

Clearly, welfare losses occur in the high-price and gap equilibrium regions: if all firms were forced to, for example, set some common price weakly below v_L, the mass $(1-\alpha)^n$ of consumers without a full match at any firm would purchase (deterministically instead of probabilistically in the gap-equilibrium region), creating an additional surplus of v_L for each additional consumer served. Moreover, the aggregate search friction would not be affected, since all consumers would still find it optimal to search until they find a full match. We may hence state the following.

Proposition 5. *In the high-price equilibrium ($\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \le \underline{\gamma}$), a deterministic welfare loss of $v_L(1-\alpha)^n$ occurs, relative to a situation where firms cannot price above v_L. In the* *gap equilibrium ($\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$), an expected welfare loss of $v_L(1-\alpha)^n(1-\kappa)^n$ occurs, relative to a situation where firms cannot price above v_L.*[22]

We now turn to comparative statics. Apart from the gap equilibrium, it can easily be seen that the comparative statics of W with respect to the model parameters are monotonic in each equilibrium region: welfare is strictly increasing in v_H and α, weakly increasing in v_L and n (with strict inequality for v_L when not in the high-price region, and strict inequality for n when not in the Bertrand region), and strictly decreasing in s.[23] All of these results are intuitive, as only direct effects are at play: a higher v_H, v_L and α lead to higher (expected) match values realized, a higher n introduces more product variety to generate additional full matches, while a higher s leads to a higher total search friction incurred.

Surprisingly, it turns out that these intuitive comparative statics do not necessarily extend to the gap-equilibrium region. In particular, we can prove analytically[24] that for $n = 2$, there exists an, albeit small, parameter region, close to the boundary to the high-price equilibrium region, where the comparative statics of welfare with respect to v_H and s flip: there, somewhat paradoxically, a marginally higher v_H decreases social welfare, while a marginally higher s increases it. The reason is, that in this specific region, the direct positive effect of a higher v_H or a lower s on welfare is more than outweighed by an indirect negative strategic effect, as firms shift probability mass to prices above v_L in response (κ decreases). Moreover, for $n = 2$, we can show numerically that the same is (sometimes) true for increases in α: close to the boundary to the high-price equilibrium region, marginal increases in α may, but need not, decrease social welfare.[25]

For (discrete) changes in n, what actually matters when assessing the induced change of welfare is how the probability that *at least one* firm prices below v_L, $1-(1-\kappa)^n$, is affected. We have checked, again numerically, that this probability may indeed decrease when n increases. However, even though such a negative strategic effect on welfare through

[22] Note that since prices need to be non-negative, if firms cannot price above v_L, the maximal distance between any two prices is v_L. Even at this maximal distance, consumers would still keep searching a firm with price v_L after having only obtained partial matches if it holds that $v_L \leq v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). One can check that this is automatically satisfied in the high-price equilibrium region, while it only holds if v_L is not too large in the gap-equilibrium region. If v_L is large in the gap-equilibrium region, the second statement in the proposition is only precise if firms' prices do not lie too far apart.

[23] Note that welfare in the low-price region is strictly increasing in α and n since $v_H - \frac{s}{\alpha} > v_L$ in this region.

[24] Details are available from the authors upon request.

[25] For $n \geq 3$, we have not been able to find any (numerical) examples in which the described counterintuitive comparative statics with respect to v_H, s or α prevail.

an increased deadweight loss may occur, our numerical simulations suggest that the positive direct effect through larger product variety always dominates.[26]

Finally, for changes in v_L, the direct and indirect welfare effects point in the same direction, as also firms' equilibrium pricing becomes more aggressive due to decreased production differentiation when v_L increases.

Combining these findings with the fact that welfare is continuous *across* the equilibrium regions, which can easily be verified, enables us to state the following.

Proposition 6. *Total social welfare weakly increases in v_L, and strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$. Moreover, apart from the gap-equilibrium region, social welfare strictly increases in v_H and α, weakly increases in n (and strictly so for $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \bar{\gamma}$), and strictly decreases in s. In the gap-equilibrium region, welfare may be locally decreasing in v_H and α, and locally increasing in s.*

We next consider the comparative statics of firm profits (and producer surplus). For this, effectively only two cases need to be considered, as firms' profits are zero in the Bertrand region, and functionally identical in the low-price and gap equilibrium. Hence, only the low-price/gap equilibrium profits and the high-price equilibrium profits remain, both available in closed form. Inspection of π_H^* (see equation (6)) and π_L^* (see equation (12)) then immediately reveals that firms' expected profits are, whenever they are positive, strictly increasing in v_H, weakly decreasing in v_L (strictly so when not in the high-price region), and strictly decreasing in s. Intuitively, a higher v_H increases product differentiation and the surplus offered to fully-matched consumers, relaxing competition and allowing firms to choose higher prices. A higher v_L intensifies competition by decreasing product differentiation, but also allows the lowest-priced firm to extract more surplus from those consumers who do not have a full match anywhere. However, the former effect always dominates. A higher s depresses prices and profits as consumers become more picky when to search on higher-priced firms, intensifying competition.

[26] Unfortunately, we have not been able to prove this analytically.

Next, as should be expected, increased competition through higher n decreases expected firm profits.[27] Remarkably, the showcased model differs from many standard oligopoly models in the sense that the aggregate firm profits (i.e, producer surplus) may also *increase* in n. Intuitively, this is true because for a relatively low probability of full matches α and a low initial number of firms, the market may expand considerably with entry, as a large number of new consumers (who did not have a full match at any firm previously) may be served (at relatively high prices above v_L).[28]

Finally, it is straightforward to check that firm profits behave non-monotonically in the probability of full matches α. For low values of α, there is Bertrand-type competition, as full matches are so unlikely that consumers do not find it worthwhile to search on after sampling the lowest-priced firm. Starting from a critical threshold, $\alpha \geq \frac{s}{v_H - v_L}$,[29] firms gain market power with increases in α, since there is scope to set prices above marginal cost that will still trigger search by consumers with only partial matches at all lower-priced rival firms (if any). However, as α increases further, product differentiation and firm profits start to decrease again. This is because, as α approaches 1, most consumers will have a full match early on in their search path, leading to strong price competition and low profits as firms attempt to be sampled early by consumers. For $\alpha = 1$, once again Bertrand competition results. Proposition 7 summarizes our findings.

Proposition 7. *Suppose firms make positive profits, $\frac{v_L}{v_H} < \bar{\gamma}$. Then individual and aggregate firm profits strictly increase in v_H, weakly decrease in v_L,[30] and strictly decrease in s. They are ambiguous in α. At the same time, individual firm profits strictly decrease in n, while aggregate firm profits are ambiguous in it.*

[27] In the high-price region, this is obvious. In the gap- and low-price regions, this can be seen directly by rewriting

$$\pi_L^* = \frac{\left[(1-\alpha)^n + \alpha\right]\left[\alpha(v_H - v_L) - s\right]}{1 - 2\alpha + \frac{\alpha}{(1-\alpha)^{n-1}}},$$

for which the nominator strictly decreases and the denominator strictly increases in n.

[28] It is easiest to see this in the high-price equilibrium (which, for any given number of firms, is played if $\alpha > \frac{s}{v_H}$ and v_L is sufficiently close to 0). There, the aggregate expected producer surplus with $n+1$ firms, $\Pi_H^*(n+1) = (n+1)\left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^n\alpha$, strictly exceeds the aggregate expected producer surplus with n firms, $\Pi_H^*(n) = n\left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1}\alpha$, if and only if $(n+1)(1-\alpha)^n > n(1-\alpha)^{n-1}$, which is equivalent to $\alpha < \frac{1}{n+1}$.

[29] This condition is equivalent to $v_L/v_H \leq \bar{\gamma}$.

[30] Strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$.

We conclude this section by briefly considering the comparative statics of consumer surplus. Clearly, despite firms' mixed-strategy pricing, the consumer surplus in the market can easily be obtained indirectly by subtracting the aggregate expected firm profits from total social welfare in each of the different equilibrium regions. Interestingly, it turns out that consumer surplus is ambiguous in v_H, s, α and n.[31] The reason is that, while increases in v_H, α and n or a decrease in s increase consumer surplus for *fixed* prices,[32] this positive effect may be more than offset when firms strategically respond by raising prices. Only for the parameter v_L it can readily be established that consumer surplus always weakly increases in it, and strictly so when not in the high-price equilibrium region.[33] This is because for larger v_L, firms' equilibrium pricing becomes unambiguously more aggressive, while those consumers who do not find a full match anywhere (may) obtain a higher partial match. We formally state these results in Proposition 8.

Proposition 8. *Consumer surplus weakly increases in v_L (and strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$), while it is ambiguous in v_H, s, α and n.*

4 Price-Directed Search and Collusion

Having set out our baseline model, we now turn to the question how firm coordination could affect market outcomes. As argued in the Introduction, collusive behavior should be particularly likely to emerge in the price-transparent markets we consider, as firms can easily monitor their rivals and promptly discipline deviators.

We proceed as follows. First, in Subsection 4.1, we examine tacit collusive agreements. We characterize the payoff-dominant symmetric collusive candidate equilibria across the parameter space and outline their welfare implications. Second, in Subsection 4.2, we con-

[31]For v_H, s and α, this is easily obtained. While numerical simulations suggest that consumer surplus typically rises in n, parameter constellations can be found where it decreases slightly when going from $n = 2$ to $n = 3$. Details are available from the authors upon request.

[32]For an increase in n, this is (for instance) true if one assumes that the newly introduced firm chooses the same price as the highest-priced incumbent. In any case, it is clear that introducing an additional firm while keeping the prices of incumbents fixed cannot decrease consumer surplus.

[33]To see this, recall that total social welfare weakly increases in v_L everywhere (compare with Proposition 6), while aggregate firm profits weakly decrease in v_L everywhere (compare with Proposition 7), and in the former case, this is strict when not in the high-price region.

sider the optimal strategy of an all-inclusive cartel maximizing industry profit, and study its effects on welfare as well.

4.1 Tacit Collusion

We now analyze tacit collusive pricing schemes. We focus on all-inclusive, symmetrical and payoff-dominant collusive pricing sustained by firms using a "grim-trigger" strategy.[34] Whenever we refer to a collusive agreement being "optimal", it is in this specific class of collusive schemes.

Our setup here is an infinitely repeated game in which each stage corresponds to the static game described above. Further, we assume that there is a "new" unit mass of consumers at every stage of the repeated game, and that all "old" consumers leave the market, even if some of them have not been served. Clearly, this excludes anticipation of firms' future pricing on consumers' part, which could in turn influence decisions on current-stage purchases and equilibrium pricing. We also assume that firms evaluate future profits according to a common discount factor $\delta \in (0,1)$.

First, we show that, depending on the market fundamentals, firms would like to coordinate on different optimal collusive prices.

Proposition 9. *The payoff-dominant, symmetric collusive price is given by*[35]

$$p^C = \begin{cases} \alpha v_H + (1-\alpha)v_L - s \leq v_L & \text{if } \alpha \leq \frac{s}{v_H}, \text{ or } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \geq \bar{\gamma} \\ v_L & \text{if } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \in [\gamma^C, \bar{\gamma}) \\ v_H - \frac{s}{\alpha} > v_L & \text{if } \alpha > \frac{s}{v_H} \text{ and } \frac{v_L}{v_H} \in [0, \gamma^C), \end{cases} \tag{20}$$

where

$$\gamma^C := \left(1 - \frac{s}{\alpha v_H}\right)[1-(1-\alpha)^n] \in (\underline{\gamma}, \bar{\gamma}). \tag{21}$$

[34] We can easily characterize also other strategy combinations to enforce cooperation, such as "stick-and-carrot" strategies (i.e., optimal penal codes, see Abreu (1988)). Such more complex strategies are able to sustain collusion on the optimal price level for a larger set of discount factors δ. However, they clearly make no difference on the equilibrium path whenever collusion can also be sustained with grim-trigger strategies.

[35] In what follows, we use the tie-breaking rule that firms coordinate on the price level which induces higher demand in case they are indifferent between two price levels. This implies that for $\frac{v_L}{v_H} = \gamma^C$, firms coordinate on v_L.

Collusion on this price level can be sustained using grim-trigger strategies if and only if firms' discount factor δ *is sufficiently close to* 1.

Proof. See Appendix A. □

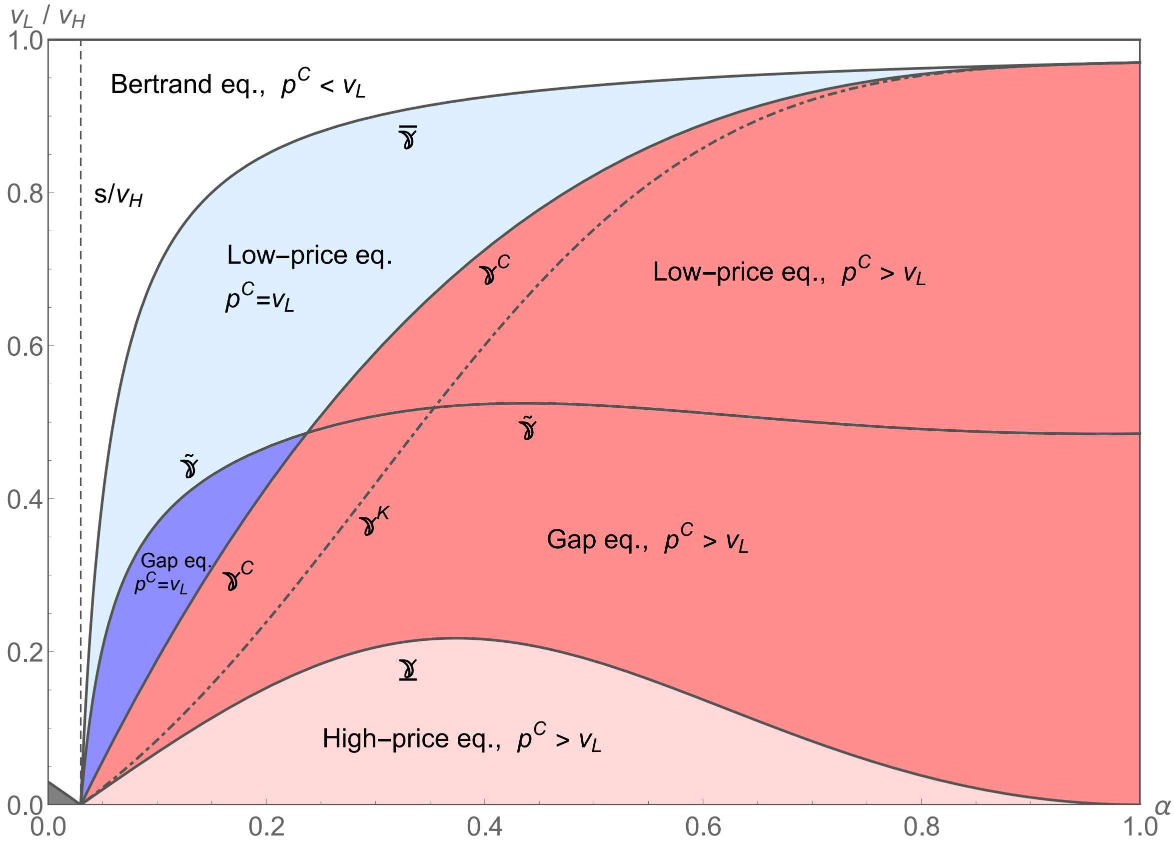

Figure 3: Pricing and welfare comparison between the optimal symmetric collusive schemes and the equilibria of the baseline model. The parameters used are $\frac{s}{v_H} = 0.03$ and $n = 3$.

Figure 3 provides a graphical representation of the different optimal collusive schemes partitioning the parameter-space (it also contains a welfare comparison to the baseline model which will be explained below). For now, in the white region above $\bar{\gamma}$, the optimal symmetric collusive price is $p^C = \alpha v_H + (1-\alpha)v_L - s < v_L$, in the light blue *and* dark blue regions between γ^C and $\bar{\gamma}$, the optimal price is $p^C = v_L$, while in the light red *and* dark red regions below γ^C, the optimal price is $p^C = v_H - \frac{s}{\alpha} > v_L$.

We will now explain the intuition behind these different cases and their welfare implications. Recall first that if full matches are quite unlikely, or when consumers' valuations of full and partial matches are relatively close ($\alpha \leq s/v_H$, or $\alpha > s/v_H$ and $v_L/v_H \geq \bar{\gamma}$), firms face a Bertrand-type competition. Consumers search at most once, and only for prices below v_L, then buy regardless of whether a full or partial match is discovered. In this case, the optimal collusive agreement involves all firms charging a price corresponding to the expected gross utility of a consumer searching exactly one firm ($\alpha v_H + (1-\alpha)v_L - s$). Still, the whole market is served, such that no deadweight loss results from collusion. The only consequence is a redistribution of rents from consumers to firms (white region in Figure 3).

When not in the Bertrand region, the highest price which firms can coordinate on is the highest price which keeps consumers in the market, $p_{max} = v_H - \frac{s}{\alpha} > v_L$. Firms should optimally set this price when consumers' valuation for partial matches is relatively low, $v_L/v_H < \gamma^C$. The consequences for welfare are clear. Whenever, in the corresponding equilibrium of the baseline model, there is a positive probability of prices being so low that consumers with only partial matches at every firm are served, $v_L/v_H \in (\underline{\gamma}, \gamma^C)$, the collusive agreement leads to a deadweight welfare loss (and a redistribution of consumer rents to firms). This is because the collusive price lies above v_L deterministically (dark-red region in Figure 3). Otherwise, for $v_L/v_H \leq \underline{\gamma}$, the consequence of collusion is, again, a mere redistribution of consumer rents (light-red region in Figure 3).

However, when with $v_L/v_H \in [\gamma^C, \bar{\gamma})$ product heterogeneity is intermediate (partial matches are neither very low compared to full matches, but also not almost as high) – or alternatively, for fixed v_L/v_H, the probability of full matches is intermediate[36] – it is no longer most profitable for firms to coordinate on a price higher than v_L. This is because by excluding consumers who do not find a full match at any firm, substantial revenue losses would be incurred. Firms instead optimally coordinate on the highest price level that is low enough to guarantee that the whole market is served ($p^C = v_L$). This obviously has no effect on total welfare when all firms also set such low prices with probability one in the corresponding one-shot Nash equilibrium, which holds if $v_L/v_H \geq \tilde{\gamma}$. In this case, consumer rents are again clearly transfered to firms (light-blue region in Figure 3).

But, as one of our main findings, it can be seen that a collusive coordination on v_L can also be optimal when firms set prices below v_L only with positive probability in the equi-

[36]Note that $\gamma^C(\alpha)$ is strictly increasing in α over the relevant range, with $\gamma^C(\frac{s}{v_H}) = 0$ and $\gamma^C(1) = 1 - \frac{s}{v_H}$. Moreover, $\gamma^C < \bar{\gamma}$. Hence, for any $v_L/v_H \in (0, 1 - \frac{s}{v_H})$, colluding on $p^C = v_L$ will be optimal if and only if α is intermediate (compare with Figure 3). An alternative condition for optimal collusion on v_L is therefore that the probability of full matches needs to be relatively low, but not so low to end up in the Bertrand region.

librium of the one-shot game (the gap equilibrium), which is the case for $v_L/v_H \in [\gamma^C, \tilde{\gamma})$, thereby *reducing* deadweight loss. The overall implication is that welfare in such markets may actually *increase* when firms collude (dark-blue region in Figure 3).[37] We moreover show that for any number of firms, there is a parameter region where this is the case. Proposition 10 formally summarizes our findings.

Proposition 10. *Suppose that* $\alpha > \frac{s}{v_H}$.[38] *Compared to the baseline model, when the optimal collusive price level can be supported in equilibrium, total social welfare remains constant and consumer rents are redistributed to firms for* $\frac{v_L}{v_H} \leq \underline{\gamma}$ *or* $\frac{v_L}{v_H} \geq \max\{\tilde{\gamma}, \gamma^C\}$. *Welfare and consumer surplus strictly decrease for* $\frac{v_L}{v_H} \in (\underline{\gamma}, \gamma^C)$. *Welfare strictly increases for* $\frac{v_L}{v_H} \in [\gamma^C, \tilde{\gamma})$, *while consumer surplus may decrease or increase. For any number of firms, there exists a parameter region where* $\tilde{\gamma} > \gamma^C$, *such that welfare may indeed increase through collusion.*

Proof. See Appendix A. □

To gain some intuition why the payoff-dominant symmetric collusive equilibrium may increase welfare, relative to the one-shot Nash equilibrium, consider a situation where with $\alpha > s/v_H$ product differentiation can be sufficiently large (for v_L sufficiently small) such that there is not always Bertrand pricing in the baseline equilibrium. Recall then that there are two conditions for collusion to increase welfare. First, product differentiation needs to be sufficiently high, $v_L/v_H < \tilde{\gamma}$, such that with unrestrained competition, firms would not always price below v_L – otherwise, welfare clearly cannot improve through collusion. For α sufficiently close above s/v_H, it can now be shown that a higher α, by giving firms scope to price above marginal cost, relaxes this condition. But second, v_L also needs to be sufficiently large, $v_L/v_H \geq \gamma^C$, such that firms find it optimal to collude on v_L rather than on the highest possible price $v_H - \frac{s}{\alpha} > v_L$. This second condition becomes harder to satisfy as α increases, as both $v_H - \frac{s}{\alpha}$ and the corresponding demand $1-(1-\alpha)^n$ strictly increase in α. Still, for values of α that are not too far above s/v_H, there may exist a range of v_L/v_H such that both conditions are satisfied. In particular, we can show that this is always true for search costs that are sufficiently close to zero. If this is the case, collusion indeed increases welfare.

[37] We can even find parameter constellations in this region where also consumer rents increase in face of firms' coordination. However, this only seems to be possible for duopoly, and only if α and v_L/v_H are very small. Details can be obtained from the authors upon request.

[38] For $\alpha \leq \frac{s}{v_H}$, total welfare remains constant, and only consumer rents are transferred to firms.

As an additional result, note that if product differentiation is not too large (with $v_L/v_H \geq \gamma^C$) and firms already optimally collude on v_L, a small decrease in search costs may actually lead to a significant *decrease* in total welfare. This is because a decrease in s makes collusion on the highest possible price level $v_H - \frac{s}{\alpha}$ relatively more attractive than colluding on v_L, such that the optimal collusive price may switch to the former. Welfare then decreases if the drop in aggregate search costs through lower s is more than offset by the additional deadweight loss caused by non-fully-matched consumers dropping out of the market. In particular, a marginal decrease of s starting from $v_L/v_H = \gamma^C$ decreases welfare by $v_L(1-\alpha)^n$, such that by continuity, also a sufficiently small *discrete* decrease from s to $s' < s$, starting from v_L/v_H close above $\gamma^C(s)$ and leading to $v_L/v_H < \gamma^C(s')$, will decrease welfare.

4.2 Cartelization

We finally consider firms joining an all-inclusive cartel (or alternatively, merging into one multi-product retailer). We assume that the total cartel profits are divided equally among its members. As we will show, the profit-maximizing strategy of such a cartel can be different from some symmetric collusive agreements, generating a higher industry profit and leading to different welfare implications.

But clearly, in the Bertrand region, where either $\alpha \leq s/v_H$, or $\alpha > s/v_H$ and $v_L/v_H \geq \bar{\gamma}$, a cartel's profit-maximizing strategy coincides with the optimal symmetric collusive agreement, since consumers only ever search once, and the highest price that induces search is given by $\alpha v_H + (1-\alpha)v_L - s \leq v_L$.[39] Similarly, when not in the Bertrand region, and if products are sufficiently differentiated with v_L/v_H sufficiently small, the cartel will, like in the optimal symmetric collusive agreement, find it most profitable to set the highest possible price that still induces search for every product, thus losing out on consumers who do not discover a full match at any firm. If so, the profit-maximizing strategy of the cartel is unique, and prices are identical to those in Proposition 9 (each cartel member prices at $v_H - \frac{s}{\alpha} > v_L$). We show in the proof of the subsequent Proposition 11 that this is the case if and only if $v_L/v_H < \gamma^K := \left(1 - \frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha+(1-\alpha)^n}$.

[39] However, the associated cartel profit could also be achieved differently, e.g. by having only *one* firm set the highest possible price which induces search by all consumers, while the other firms could set arbitrarily higher prices.

However, if the ratio of consumers' valuations takes on intermediate values, $v_L/v_H \in [\gamma^K, \bar{\gamma})$ – or, alternatively, if the probability of full matches α is intermediate – it is optimal to charge the low price v_L for *exactly one* product and the highest possible price $v_H - \frac{s}{\alpha}$ that keeps consumers in the market for all other products.[40] By providing the former "compromise option", firms extract as much consumer rent as possible while *still* serving all consumers who do not find a full match at any firm. This is particularly attractive if full matches are relatively rare. Obviously, setting the price of more than one product to v_L cannot be optimal, since total demand would be unchanged, but a larger fraction of consumers would buy at this low price.[41]

Note that if the cartel finds it most profitable to serve the whole market by pricing one product at v_L, the same implications for welfare hold as in Proposition 10 when $p^C \leq v_L$. Hence, welfare remains constant whenever the whole market would be served in the equilibrium of the baseline game as well, while it strictly increases otherwise. However, it is easily shown that the parameter region in which total welfare increases is strictly larger (in the sense of set inclusion) than under the optimal symmetric collusive scheme.[42] Intuitively, whenever it is the most profitable *symmetric* collusive strategy to serve the whole market at a price of v_L, rather than to serve only part of the market at $v_H - \frac{s}{\alpha} > v_L$, setting the price of only one product to v_L (such that all consumers who do not find a full match at any firm stay in the market) while charging the highest possible price for all other products must be even more profitable in comparison – the strategy increases the average mark-up without losing out on demand. Hence, the range of parameters where firms find it optimal to price one product at v_L (rather than to collectively price at $v_H - \frac{s}{\alpha} > v_L$) is larger than the range of parameters where firms find it optimal to symmetrically collude at v_L (again, rather than to collectively price at $v_H - \frac{s}{\alpha} > v_L$). In Figure 3, one can observe that the region where

[40]Similar to the reasoning in Footnote 36 above, note that $\gamma^K(\alpha)$ is strictly increasing in α over the relevant range, with $\gamma^K(\frac{s}{v_H}) = 0$ and $\gamma^K(1) = 1 - \frac{s}{v_H}$. Moreover, $\gamma^K < \bar{\gamma}$. Hence, for any $v_L/v_H \in (0, 1 - \frac{s}{v_H})$, it is optimal for the cartel to price one product at v_L and all others at $v_H - \frac{s}{\alpha}$ if and only if α is intermediate (compare once more with Figure 3). Hence, the probability of full matches needs to be relatively low, but not so low to end up in the Bertrand region.

[41]Let $1 \leq m \leq n$ products' prices be at v_L, and the remaining $n-m$ prices at $v_H - \frac{s}{\alpha}$. Then, since only those consumers who have no full match at any of the m low-priced firms *and* have a full match at some of the $n-m$ high-priced firms buy from the latter, while all others buy from the former, the cartel makes a profit of $\pi = \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^m[1-(1-\alpha)^{n-m}] + v_L[1-(1-\alpha)^m[1-(1-\alpha)^{n-m}]] = v_L + \left(v_H - \frac{s}{\alpha} - v_L\right)[(1-\alpha)^m - (1-\alpha)^n]$. Since in the "non-Bertrand" region $v_H - \frac{s}{\alpha} > v_L$, profits are clearly maximal for $m = 1$.

[42]The statement boils down to $\gamma^C > \gamma^K \Leftrightarrow 1-(1-\alpha)^n > \frac{\alpha}{\alpha+(1-\alpha)^n} \Leftrightarrow 1-\alpha-(1-\alpha)^n > 0$, which is true.

welfare increases through firm coordination would expand from the dark-blue region to the region between γ^K and $\tilde{\gamma}$. We now state our findings formally.

Proposition 11. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} < \bar{\gamma}$.[43] *Then, if* $\frac{v_L}{v_H} \in [0, \gamma^K)$*, where* $\gamma^K := \left(1 - \frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha+(1-\alpha)^n} \in (\underline{\gamma}, \gamma^C)$*, the all-inclusive cartel's profit-maximizing strategy has all firms setting their price at* $v_H - \frac{s}{\alpha}$*. If instead* $\frac{v_L}{v_H} \in [\gamma^K, \bar{\gamma})$*, the all-inclusive cartel's profit-maximizing strategy involves firms setting prices such that*

$$p_i^K := v_L \qquad \textit{for one } i \in \{1, \ldots, n\}$$
$$p_j^K := v_H - \frac{s}{\alpha} \qquad \forall j \neq i.$$

For $\frac{v_L}{v_H} \in [\gamma^K, \gamma^C)$*, total social welfare and consumer surplus strictly increase, compared to the payoff-dominant collusive equilibrium. For* $\frac{v_L}{v_H} \in [\gamma^K, \tilde{\gamma})$*, total social welfare strictly increases under cartelization, compared to the non-cooperative outcome.*

Proof. See Appendix A. □

In principle, we could also allow firms to choose the above asymmetric strategy combination without forming a cartel, but tacitly coordinating on this behavior might arguably be substantially more difficult to sustain.

5 Conclusion

We have set up a tractable model of price-directed search in which consumers observe prices, but need to engage in costly sequential search in order to find out whether products fully or only partially match their needs. We have characterized the set of symmetric equilibria and show that welfare losses may occur, as all firms may (deterministically or stochastically) price above consumers' valuation for partial matches. If this happens, part of the consumers inefficiently drop out of the market. Analyzing collusion and cartelization, we find that, perhaps surprisingly, social welfare may in fact increase in face of such coordination. This is particularly likely when search costs are low, products are sufficiently differentiated, and

[43] If $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \bar{\gamma}$, the cartel solution has all firms pricing at $p^K = \alpha v_H + (1-\alpha)v_L - s \leq v_L$, welfare is unaffected relative to the baseline game, while all consumer surplus is transferred to firms.

consumers have moderately “picky” tastes such that the likelihood of full matches is not too high. One potential implication for policymakers is that firm coordination on relatively low prices, in particular in (online) search markets where consumers derive some baseline utility from products but the incidence of very good matches is relatively low, may be treated more benevolently, as unrestricted competition may even lead to worse market outcomes.

References

Dilip Abreu. On the Theory of Infinitely Repeated Games with Discounting. *Econometrica*, 56(2): 383–396, March 1988.

Simon P. Anderson and Régis Renault. Pricing, Product Diversity, and Search Costs: A Bertrand-Chamberlin-Diamond Model. *RAND Journal of Economics*, 30(4):719–735, Winter 1999.

Simon P. Anderson and Régis Renault. Firm pricing with consumer search. In *Handbook of Game Theory and Industrial Organization, Volume II*, Chapters, chapter 8, pages 177–224. Edward Elgar Publishing, February 2018.

Maria Arbatskaya. Ordered search. *RAND Journal of Economics*, 38(1):119–126, March 2007.

Mark Armstrong. Ordered Consumer Search. *Journal of European Economic Association*, 15(5): 989–1024, 2017.

Mark Armstrong and Jidong Zhou. Paying for prominence. *The Economic Journal*, 121:368–395, 2011.

Mark Armstrong, John Vickers, and Jidong Zhou. Prominence and consumer search. *RAND Journal of Economics*, 40(2):209–233, 2009.

Susan Athey and Glenn Ellison. Position auctions with consumer search. *Quarterly Journal of Economics*, 126:1213–1270, 2011.

Michael Baye, J. Rupert J. Gatti, Paul Kattuman, and John Morgan. Clicks, discontinuities, and firm demand online. *Journal of Economics & Management Strategy*, 18(4):935–975, 2009.

Michael R. Baye, John Morgan, and Patrick Scholten. Information, Search, and Price Dispersion. In *Handbook in Economics and Information Systems (ed. T. Hendershott)*, volume 1, chapter 6. Amsterdam: Elsevier, 2006.

Colin Campbell, Gautam Ray, and Waleed A. Muhanna. Search and Collusion in Electronic Markets. *Management Science*, 51(3):1497–1507, 2005.

Yongmin Chen and Chuan He. Paid placement: advertising and search on the internet. *The Economic Journal*, 121:F309–F328, 2011.

Michael Choi, Anovia Y. Dai, and Kyungmin Kim. Consumer Search and Price Competition. *Econometrica*, 86(4):1257–1281, 2018.

Yucheng Ding and Tianle Zhang. Price-directed consumer search. *International Journal of Industrial Organization*, 58:106–135, 2018.

Glenn Ellison and Sara Fisher Ellison. Search, obfuscation, and price elasticities on the internet. *Econometrica*, 77(2):427–452, 2009.

Switgard Feuerstein. Collusion in Industrial Economics–A Survey. *Journal of Industry, Competition and Trade*, 5(3):163–198, December 2005.

Marco Haan, José L. Moraga-González, and Vaiva Petrikaite. A Model of Directed Consumer Search. *International Journal of Industrial Organization*, 61:223–255, 2018.

Norman J. Ireland. The Provision of Information in a Bertrand Oligopoly. *Journal of Industrial Economics*, 41(1):61–76, March 1993.

José L. Moraga-González. Consumer Search in Digital Markets. Survey, Vrije Universiteit Amsterdam and University of Groningen, 2018.

José L. Moraga-González, Zsolt Sandor, and Matthijs Wildenbeest. Consumer Search and Prices in the Automobile Market. CEPR Discussion Papers 10487, C.E.P.R. Discussion Papers, March 2015. URL `https://ideas.repec.org/p/cpr/ceprdp/10487.html`.

Arvid Nilsson. Transparency and Competition. SSE/EFI Working Paper Series in Economics and Finance 298, Stockholm School of Economics, February 1999. URL `https://ideas.repec.org/p/hhs/hastef/0298.html`.

Mitsukuni Nishida and Marc Remer. Lowering consumer search costs can lead to higher prices. *Economics Letters*, 162(C):1–4, 2018.

Vaiva Petrikaitė. Collusion with costly consumer search. *International Journal of Industrial Organization*, 44(C):1–10, 2016.

Christian Schultz. Transparency on the consumer side and tacit collusion. *European Economic Review*, 49(2):279–297, February 2005.

Christian Schultz. Collusion in Markets with Imperfect Price Information on Both Sides. *Review of Industrial Organization*, 50(3):287–301, May 2017.

Jian Shen. Ex-Ante Preference in a Consumer Search Market. Technical report, SSRN, April 2015. URL `https://ssrn.com/abstract=2826011`.

Dale O. Stahl. Oligopolistic Pricing with Sequential Consumer Search. *American Economic Review*, 79(4):700–712, September 1989.

Hal R. Varian. A Model of Sales. *American Economic Review*, 70(4):651–59, September 1980.

Asher Wolinsky. True Monopolistic Competition as a Result of Imperfect Information. *The Quarterly Journal of Economics*, 101(3):493–511, August 1986.

Jidong Zhou. Ordered search in differentiated markets. *International Journal of Industrial Organization*, 29(2):253–262, March 2011.

6 Appendix A: Technical Proofs

Proof of Proposition 2. We first give a detailed existence proof. We then provide a sketch how uniqueness can be established.

Existence. Given that all other firms sample prices from the CDF $F_H(p)$ as defined in equation (3), it is first easy to see that for any price in the candidate equilibrium's support $[\underline{p}_H, \overline{p}_H]$, it indeed holds that $\pi_i(p) = p(1-\alpha F_H(p))^{n-1}\alpha = \pi_H^*$, with π_H^* as defined in equation (6). It is moreover straightforward to check that given the imposed parameter restrictions $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in [0, \underline{\gamma}]$, $F_H(p)$ is strictly increasing in its support, and that $\underline{p}_H > v_L$. Hence, the candidate equilibrium is well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. We start by proving that there are no profitable deviations below the lowest price $\underline{p}_H$. For this, note first that it clearly cannot be optimal to deviate to any price in the range $(v_L, \underline{p}_H)$, as the same demand would already be achieved when pricing at $\underline{p}_H$. Note next that when deviating to v_L, a firm would make an expected profit of $\pi_i(v_L) = v_L[\alpha + (1-\alpha)^n]$, as it would become the lowest-priced firm that is sampled first with certainty, attracting all of its fully-matched consumers as well as all consumers with no full match at any firm (who would eventually return to the deviating firm after having searched all firms). Note moreover that those consumers who are only partially matched at the deviating firm would always continue to search, since even if all rival firms priced at $\overline{p}_H$, the expected gains from search would be non-negative. Given that $\alpha > \frac{s}{v_H}$ as assumed, it is then easy to see that $\pi_i(v_L) \leq \pi_H^*$ if and only if $\frac{v_L}{v_H} \leq \underline{\gamma}$, as also assumed.

We next establish that under the relevant parameter restrictions, it is never profitable to price below v_L, as the expected profits for any deviation price $p \in (0, v_L)$ are lower than when deviating to v_L. To see this, note that since the deviating firm is guaranteed to be searched first, the fraction α of consumers who find a full match at this firm will immediately buy there. Furthermore, consumers who only find a partial match will only search those rival firms j (and buy there in case they find a full match) whose price difference is not too large relative to the deviation price, that is, for which $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). The probability that one rival sets $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (such that it will be searched) and provides a full match (such that it will attract the deviating firm's partially-matched consumers) is given by $F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha$. Hence, the probability that *not a single* rival firm does so is given by $\left[1 - F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}$. In turn, the expected deviation profits for $p \leq v_L$ can be written as

$$\begin{aligned}\pi_i(p) &= p\left[\alpha + (1-\alpha)\left[1 - F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right] \\ &= p\left[\alpha + (1-\alpha)^n\left(\frac{v_H - \frac{s}{\alpha}}{p + v_H - v_L - \frac{s}{\alpha}}\right)\right].\end{aligned}$$

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p when $v_H - v_L - \frac{s}{\alpha} > 0$ (as holds in the considered parameter region), it is easy to see that the last expression is strictly increasing in p. It is thus indeed maximized for $p = v_L$, such that deviations below v_L cannot be optimal.

It remains to show that there is no profitable deviation above $\overline{p}_H = v_H - \frac{s}{\alpha}$. But since no firm would ever be searched for $p > \overline{p}_H$ (compare once again with Lemma 1), this is immediately evident. This completes the proof of existence.

Uniqueness. For brevity, we only provide a sketch how uniqueness can be established in the class of symmetric equilibria. This sketch also applies for the subsequent Propositions 3 and 4.

Note first that the parameter requirement for Proposition 2 (as well as Propositions 3 and 4) is that $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \overline{\gamma}$, which is equivalent to $\alpha(v_H - v_L) > s$. Only in this case, consumers may have an incentive to search on after discovering only a partial match at the lowest-priced firm, avoiding the Bertrand outcome as unique symmetric equilibrium.

Second, since the parameter requirement $\alpha(v_H - v_L) > s$ is equivalent to $v_H - \frac{s}{\alpha} > v_L$, it follows immediately from consumers' optimal search rule in Lemma 1 that no firm can make a positive profit when pricing strictly above $p_{max} := v_H - \frac{s}{\alpha}$, as it would never be searched. But clearly, each firm can guarantee a positive profit by pricing at $v_H - v_L - \frac{s}{\alpha} > 0$, since it would be searched by only partially-matched consumers even for $p_1 = 0$. Hence, no firm may ever price above p_{max} in equilibrium.

Third, given that $\alpha(v_H - v_L) > s$, clearly no symmetric pure-strategy equilibrium can exist, as marginally undercutting any symmetric candidate equilibrium price $p^* \in (0, v_H - \frac{s}{\alpha}]$ would give a firm a discretely higher profit (by being searched first by all consumers). By a similar logic, there can be no mass points in any symmetric equilibrium.

Denoting $\overline{p}$ and $\underline{p}$ as the upper and lower support bound of any symmetric candidate equilibrium, with $\overline{p} \leq p_{max}$, the crucial steps are now to establish that either (i) $\overline{p} - \underline{p} < \Delta := v_H - v_L - \frac{s}{\alpha}$ and $\overline{p} = p_{max}$ or (ii) $\overline{p} - \underline{p} = \Delta$. The former is trivial to see by contradiction: in any candidate equilibrium where $\overline{p} - \underline{p} < \Delta$ and $\overline{p} < p_{max}$, a firm choosing $p_i = \overline{p}$ could unilaterally increase its profit by choosing $p_i = \min\{\underline{p} + \Delta, p_{max}\}$ instead. This gives the firm an identical demand of $(1-\alpha)^{n-1}\alpha$ at a higher price (in particular, since there can be no mass point at $\overline{p}$).

It is significantly more demanding to show that $\overline{p} - \underline{p} > \Delta$ cannot hold. This can be proven by contradiction via the following steps: (1) For $\overline{p} - \underline{p} > \Delta$, it must hold that the density $f(\overline{p}) \stackrel{!}{=} 0$ by comparing $\lim_{p\uparrow\overline{p}} \pi_i'(p)$ with $\lim_{p\downarrow\overline{p}} \pi_i'(p)$, (2) from this, it follows that the density $f(\overline{p} - \Delta) > 0$, such that $\overline{p} - \Delta$ must lie in the equilibrium support, (3) $\lim_{p\uparrow(\overline{p}-\Delta)} \pi_i'(p) = \lim_{p\downarrow(\overline{p}-\Delta)} \pi_i'(p)$ as a consequence of $f(\overline{p}) = 0$, (4) combining the conditions $\pi_i'(\overline{p}) \stackrel{!}{=} 0$ and $\pi_i'(\overline{p} - \Delta) \stackrel{!}{=} 0$[44] and finally observing that this leads to a contradiction.

Using the result that either (i) $\overline{p} - \underline{p} < \Delta$ and $\overline{p} = p_{max}$ or (ii) $\overline{p} - \underline{p} = \Delta$, the required profit indifference at $\overline{p}$ and $\underline{p}$ gives rise to a respectively unique solution for $\overline{p}$, $\underline{p}$ and the candidate equilibrium profit π^*, both for (i) (as provided in equations (4), 5) and (6)) and (ii) (as provided in equations (12), (13) and (14)). The corresponding $\overline{p}$ for (ii) is however not compatible with $\overline{p} \leq p_{max}$ if $\frac{v_L}{v_H} \leq \underline{\gamma}$,[45] as

[44] The latter must be true since $\overline{p} - \Delta$ lies in the equilibrium support, such that there must also be probability mass immediately below or above $\overline{p} - \Delta$ (or both).

[45] In the borderline case where $\frac{v_L}{v_H} = \underline{\gamma}$, it actually holds that $\overline{p} = p_{max}$ and $\underline{p} = v_L$. In particular, this would mean that the firms choose prices weakly lower than v_L with zero probability, yet $p = v_L$ lies in the support, with discretely higher demand than when setting $p + \varepsilon$ for any $\varepsilon > 0$ (due to returning demand). Hence, there would need to be a gap in the equilibrium distribution for prices slightly above v_L, which is however incompatible with $F(v_L) = 0$ and v_L being part of the support.

assumed for Proposition 2 (while it is compatible with it for $\frac{v_L}{v_H} \in (\underline{\gamma}, \overline{\gamma})$, in which case the candidate equilibrium following (i) does not exist). Noting finally that with $\overline{p} - \underline{p} \leq \Delta$ there can be no holes in the equilibrium support apart possibly from some range immediately above v_L, in each case the respective equilibrium follows uniquely from construction. □

Proof of Proposition 3. In what follows, we prove existence. For uniqueness, the argument at the end of the proof of Proposition 2 applies.

Existence. It is first easy to verify that simultaneously solving equations (7) and (8) gives $\underline{p}_L$, $\overline{p}_L$ and $\pi_L^* = \overline{p}_L(1-\alpha)^{n-1}\alpha$ as reported in the proposition. Moreover, since $\alpha(v_H - v_L) - s > 0$ in the considered parameter region, these objects are all strictly positive, with clearly $\overline{p}_L > \underline{p}_L$, and $\overline{p}_L \leq v_L$ since by assumption $v_L/v_H \geq \tilde{\gamma}$. By construction, the implicit definition of $F_L(p)$ in equation (11) ensures that all prices in the candidate equilibrium's support yield the same expected profit. One may also note from equation (11) that $F_L(p)$ is strictly increasing in its support. Hence, all equilibrium objects are well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. First, we show that there is no profitable deviation above $\overline{p}_L$. A deviating firm pricing at some $p > \overline{p}_L$ will only be searched if its price is not too high relative to the lowest-priced firm, which holds if $p_1 \geq p - \left(v_H - v_L - \frac{s}{\alpha}\right)$ (compare with Lemma 1). Equivalently, in order for the deviating firm to be searched at all, all rival firms' prices must lie above $p - \left(v_H - v_L - \frac{s}{\alpha}\right)$. Then, the deviating firm will cater to the mass $(1-\alpha)^{n-1}\alpha$ consumers who don't have a full match at any rival firm, but a full match at this firm. Thus, the expected profit at any such price $p > \overline{p}_L$ can be written as

$$\pi_i(p) = p\left[1 - F_L\left(p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha. \tag{22}$$

For prices which lie in the support of the candidate equilibrium, i.e. $p \in [\underline{p}_L, \overline{p}_L]$, the expected profit is by construction equal to π_L^*, where we replicate here the implicit definition of $F_L(p)$, equation (11), for convenience:

$$\pi_i(p) = p\left[\left(1 - \alpha F_L(p)\right)^{n-1}\alpha + \left(1 - F_L(p)\right)^{n-1}(1-\alpha)^n\right] = \pi_L^*. \tag{23}$$

Since $F_L(p)$ cannot be obtained in closed form for an arbitrary number of firms n, we will use an estimation. Rewriting (23), it holds for $p \in [\underline{p}_L, \overline{p}_L]$ that

$$(1-F_L(p))^{n-1} = \frac{\frac{\pi_L^*}{p} - (1-\alpha F_L(p))^{n-1}\alpha}{(1-\alpha)^n} \leq \frac{\frac{\pi_L^*}{p} - (1-F_L(p))^{n-1}\alpha}{(1-\alpha)^n},$$

such that by isolating $(1-F_L(p))^{n-1}$ we obtain

$$(1-F_L(p))^{n-1} \leq \frac{\pi_L^*}{p\,[\alpha+(1-\alpha)^n]}. \tag{24}$$

For $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$, it holds that $p - (v_H - v_L - \frac{s}{\alpha}) \in [\underline{p}_L, \overline{p}_L]$. Hence, by inequality (24), we have that for $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$,

$$\left[1 - F_L\Big(p - \big(v_H - v_L - \frac{s}{\alpha}\big)\Big)\right]^{n-1} \leq \frac{\pi_L^*}{\left[p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right][\alpha + (1-\alpha)^n]}.$$

In turn, this implies that the following estimation can be given for equation (22) and $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$:

$$\begin{aligned}\pi_i(p) &= p\left[1 - F_L\left(p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha \\ &\leq p\left[\frac{\pi_L^*}{\left[p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right][\alpha+(1-\alpha)^n]}\right](1-\alpha)^{n-1}\alpha.\end{aligned}$$

Since $\frac{p}{p-(v_H-v_L-\frac{s}{\alpha})}$ is strictly decreasing in p for $v_H - v_L - \frac{s}{\alpha} > 0$ as assumed for the proposition, the last expression is thereby maximized for $p = \overline{p}_L$. This implies that for $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$,[46]

$$\pi_i(p) \leq \overline{p}_L \left[\frac{\pi_L^*}{\left[\overline{p}_L - \left(v_H - v_L - \frac{s}{\alpha}\right)\right][\alpha+(1-\alpha)^n]}\right](1-\alpha)^{n-1}\alpha = \pi_L^*.$$

Hence, deviations above $\overline{p}_L$ are indeed not profitable.

Next, we show that there is no profitable deviation below $\underline{p}_L$. For such low prices, there is now a positive probability that some or all rival firms draw high enough prices such that consumers who are only partially matched at the deviating firm do not search them. Precisely, for deviation prices $p < \underline{p}_L$, consumers that are only partially matched at the deviating firm will only search rival firms j for which $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). Moreover, consumers will only buy at such firms if they are fully matched at them. The probability to lose the mass $1-\alpha$ of partially-matched consumers

[46]For $p > \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})$, $\pi_i(p) = 0$, since no consumer would ever search the deviating firm.

towards a single rival is therefore given by $F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha$. Consequently, the probability *not* to lose these consumers against *any* rival firm is given by $\Big[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\Big]^{n-1}$. Hence, we can write a deviating firm's expected profit for $p<\underline{p}_L$ as

$$\pi_i(p)=p\left[\alpha+(1-\alpha)\left[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\right]^{n-1}\right]. \tag{25}$$

Again, our strategy will be to use an estimation for the additional expected demand, which will be derived from the only implicitly defined CDF F_L. Using once more equation (23), we find that for $p\in[\underline{p}_L,\overline{p}_L]$ it holds that

$$(1-\alpha F_L(p))^{n-1}=\frac{\frac{\pi_L^*}{p}-(1-F_L(p))^{n-1}(1-\alpha)^n}{\alpha}\leq\frac{\pi_L^*}{\alpha p}. \tag{26}$$

For $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\overline{p}_L-(v_H-v_L-\frac{s}{\alpha})]=[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$, it holds that $p+(v_H-v_L-\frac{s}{\alpha})\in[\underline{p}_L,\overline{p}_L]$. Hence, by inequality (26), we have that for $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$,

$$\left[1-\alpha F_L\Big(p+\big(v_H-v_L-\frac{s}{\alpha}\big)\Big)\right]^{n-1}\leq\frac{\pi_L^*}{\alpha(p+v_H-v_L-\frac{s}{\alpha})}.$$

In turn, this implies that the following approximation can be given for equation (25) and $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$:

$$\begin{aligned}\pi_i(p)&=p\left[\alpha+(1-\alpha)\left[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\right]^{n-1}\right]\\&\leq p\left[\alpha+(1-\alpha)\left[\frac{\pi_L^*}{\alpha(p+v_H-v_L-\frac{s}{\alpha})}\right]\right].\end{aligned}$$

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p for $v_H-v_L-\frac{s}{\alpha}>0$ as assumed for the proposition, the last expression is thereby maximized for $p=\underline{p}_L$. This implies that for $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$,[47]

$$\pi_i(p)\leq\underline{p}_L\left[\alpha+(1-\alpha)\left[\frac{\pi_L^*}{\alpha(\underline{p}_L+v_H-v_L-\frac{s}{\alpha})}\right]\right]=\pi_L^*.$$

Hence, deviations below $\underline{p}_L$ are indeed not profitable. This completes the proof. □

Proof of Proposition 4. In what follows, we prove existence. For uniqueness, the argument at the end of the proof of Proposition 2 applies.

[47] For $p<\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$, $\pi_i(p)<\pi_i(\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$, since all consumers already purchase deterministically at the deviating firm for $p=\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$.

Existence. The equilibrium objects $\underline{p}_M = \underline{p}_L$, $\overline{p}_M = \overline{p}_L$ and $\pi^*_M = \pi^*_L$ originate from solving the same system of equations (7) and (8) that define $\underline{p}_L$, $\overline{p}_L$ and π^*_L. Again, they are all strictly positive since $\alpha(v_H - v_L) - s > 0$ in the considered parameter region (compare also with the proof of Proposition 3). Observe next that $\underline{p}_M < v_L$ follows from $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} > \underline{\gamma}$, while $\overline{p}_M > v_L$ follows from $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \tilde{\gamma}$, as assumed for the proposition.

To see that $\underline{p}'_M > v_L$, note the following. First, since $v_L\left[\alpha(1-\alpha\kappa)^{n-1} + (1-\kappa)^{n-1}(1-\alpha)^n\right]$ is strictly increasing in v_L for $\kappa \in [0,1]$ while π^*_L is strictly decreasing in v_L, one can clearly see via the implicit definition of $\kappa = F_{M_1}(v_L)$ in equation (17) that κ must be strictly increasing in v_L whenever $\kappa \in [0,1)$. Moreover, for $\frac{v_L}{v_H} = \underline{\gamma}$ it holds that $\kappa = 0$, while for $\frac{v_L}{v_H} = \tilde{\gamma}$, it holds that $\kappa = 1$. Hence, $\kappa \in (0,1)$ in the considered parameter region. Substituting π^*_L from equation (17) into equation (16) now yields

$$\underline{p}'_M = v_L\left[1 + \frac{(1-\alpha)^n}{\alpha}\left(\frac{1-\kappa}{1-\alpha\kappa}\right)^{n-1}\right],$$

which indeed strictly exceeds v_L for all $\kappa \in [0,1)$.

A firm's expected profit when choosing a price in the range $[\underline{p}_M, v_L]$ is given by

$$\pi_i(p) = p\left[\alpha(1-\alpha F_{M_1}(p))^{n-1} + (1-F_{M_1}(p))^{n-1}(1-\alpha)^n\right],$$

such that for $F_{M_1}(p) = F_L(p)$, it clearly holds that $\pi_i(p) = \pi^*_L = \pi^*_M$ for all prices in that interval (as follows from the implicit definition of $F_L(p)$ in equation (11)). A firm's expected profit when choosing a price in the range$[\underline{p}'_M, \overline{p}_M]$ is given by $\pi_i(p) = p\left[1 - F_{M_2}(p)\alpha\right]^{n-1}\alpha$, such that for

$$F_{M_2}(p) = \frac{1}{\alpha}\left[1 - \left(\frac{\pi^*_L}{\alpha p}\right)^{\frac{1}{n-1}}\right],$$

it also holds that $\pi_i(p) = \pi^*_L = \pi^*_M$ for all prices in that interval. It is moreover easy to see that both $F_{M_1}(p)$ and $F_{M_2}(p)$ are strictly increasing in p. Hence, all equilibrium objects are well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. First, it clearly cannot be optimal to deviate to a price $p \in (v_L, \underline{p}'_M)$, as the deviating firm would not achieve a higher expected demand than when pricing at $\underline{p}'_M > p$. When deviating to a price $p > \overline{p}_M$, the deviating firm will only be searched if all rival firms price above $p - \left(v_H - v_L - \frac{s}{\alpha}\right)$ (compare with Lemma 1). Then, the deviating firm will cater to the mass $(1-\alpha)^{n-1}\alpha$ consumers who don't have a full match at any rival firm, but a full match at this firm. Thus, the expected profit at any such price $p > \overline{p}_M$ can be written as

$$\pi_i(p) = p\left[1 - F_{M_1}\left(p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha, \tag{27}$$

where $F_{M_1}(p - (v_H - v_L - \frac{s}{\alpha}))$ is the relevant probability that a rival firm prices below $p - (v_H - v_L - \frac{s}{\alpha})$.[48] Since $F_{M_1}(\cdot) = F_L(\cdot)$, the same estimation as in the proof of Proposition 3 can now be used to show that $\pi_i(p) \leq \pi_L^* = \pi_M^*$ for all $p > \overline{p}_M$. Hence, deviations above $\overline{p}_M$ are not profitable.

We finally show that there are no profitable deviations to prices $p \in (0, \underline{p}_M)$. Following the argument in the proof of Proposition 3, a firm deviating to such a price makes an expected profit of

$$\pi_i(p) = p\left[\alpha + (1-\alpha)\left[1 - F_{M_r}\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right], \tag{28}$$

where $r = 1$ if $p + v_H - v_L - \frac{s}{\alpha} \leq v_L$ and $r = 2$ otherwise. Since $F_{M_1}(p)$ is implicitly defined by

$$p\left[\alpha(1 - \alpha F_{M_1}(p))^{n-1} + (1 - F_{M_1}(p))^{n-1}(1-\alpha)^n\right] - \pi_L^* = 0,$$

while $F_{M_2}(p)$ is implicitly defined by

$$p\left[\alpha(1 - \alpha F_{M_2}(p))^{n-1}\right] - \pi_L^* = 0,$$

it is straightforward to see that $F_{M_1}(p) > F_{M_2}(p)$ when applied for the same price. Comparing with (28), a sufficient condition to have no profitable deviations below $\underline{p}_M$ is then that for all $p \in (0, \underline{p}_M)$,

$$\pi_i(p) \leq p\left[\alpha + (1-\alpha)\left[1 - F_{M_2}\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right] \overset{!}{\leq} \pi_L^*.$$

Inserting $F_{M_2}(\cdot)$ from equation (15), the above condition is equivalent to

$$p\left[\alpha + (1-\alpha)\left[\frac{\pi_L^*}{\alpha(p + v_H - v_L - \frac{s}{\alpha})}\right]\right] \overset{!}{\leq} \pi_L^* \qquad \forall p \in (0, \underline{p}_M).$$

[48] Otherwise, $p > v_H - \frac{s}{\alpha}$, implying zero demand for the deviating firm.

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p for $v_H - v_L - \frac{s}{\alpha} > 0$ as assumed for the proposition, the LHS in the last expression is maximized for $p = \underline{p}_M = \underline{p}_L$. Hence, for $p \in (0, \underline{p}_M]$,

$$\pi_i(p) \leq \underline{p}_M \left[\alpha + (1-\alpha)\left[\frac{\pi_L^*}{\alpha(\underline{p}_M + v_H - v_L - \frac{s}{\alpha})}\right]\right] = \pi_L^*,$$

such that deviations below $\underline{p}_M$ are indeed not profitable. This completes the proof. □

Proof of Proposition 9. Note first that in the Bertrand-equilibrium region ($\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \overline{\gamma}$), the parameters are such that consumers will never search more than one firm. Given this, it is weakly optimal for a consumer to search the lowest-priced firm if it charges a weakly lower price than the expected gross utility it will provide, $p \leq \alpha v_H + (1-\alpha)v_L - s$ (compare with Lemma 1), where $\alpha v_H + (1-\alpha)v_L - s \leq v_L$ in the considered parameter region. The best firms can do is hence to coordinate their prices on $p^C = \alpha v_H + (1-\alpha)v_L - s \leq v_L$ for a maximal collusive stage-game profit of $\pi^C = \frac{\alpha v_H + (1-\alpha)v_L - s}{n}$.

If not in the Bertrand-equilibrium region, then there are only two price levels on which firms may optimally coordinate. First, for all symmetric collusive price levels $p^C \leq v_L$, all consumers will be served (randomly allocating themselves across firms in the search process), such that by pricing at v_L, each firm makes a maximal profit of $\pi^C = \frac{v_L}{n}$. Second, for all symmetric collusive price levels $p^C \in (v_L, v_H - \frac{s}{\alpha}]$, only consumers who find a full match at some firm will purchase eventually (again, randomly allocating themselves across firms in the search process).[49] The total number of such consumers is $1-(1-\alpha)^n$, for a maximal collusive per-firm profit of $\pi^C = \left(v_H - \frac{s}{\alpha}\right)\frac{1-(1-\alpha)^n}{n}$. Colluding on v_L is hence strictly more profitable if and only if

$$\frac{v_L}{v_H} > \left(1 - \frac{s}{\alpha v_H}\right)[1-(1-\alpha)^n] =: \gamma^C,$$

where it can be checked that $\gamma^C \in (\underline{\gamma}, \overline{\gamma})$ for $\alpha > \frac{s}{v_H}$,[50] as needs to hold (note that we have assumed that $\alpha < 1$ throughout).

[49] For $p^C > v_H - \frac{s}{\alpha}$, demand drops to zero.

[50] Given $\alpha > \frac{s}{v_H}$, the condition $\gamma^C > \underline{\gamma}$ is equivalent to

$$1 > (1-\alpha)^n + \alpha\left[\frac{(1-\alpha)^{n-1}}{(1-\alpha)^n + \alpha}\right],$$

which is true because $(1-\alpha)^n < 1-\alpha$ and $\frac{(1-\alpha)^{n-1}}{(1-\alpha)^n+\alpha} < 1$.

Observe finally that in each case, coordinating on the optimal collusive price level clearly gives rise to a higher stage-game profit π^C than the corresponding Nash-equilibrium profit π^N. Hence, the one-period gain when deviating optimally, π^D, will not be enough to make deviating worthwhile if firms are sufficiently patient (δ is sufficiently close to 1).[51]

□

Proof of Proposition 10. The only statement which remains to be shown is that for any number of firms, there exists a parameter region where $\tilde{\gamma} > \gamma^C$. To see this, note first that

$$\tilde{\gamma}\left(\alpha = \frac{s}{v_H}\right) = \gamma^C\left(\alpha = \frac{s}{v_H}\right) = 0.$$

By continuity of $\tilde{\gamma}$ and γ^C, it thus suffices to establish that for any $n \geq 2$, we can find parameters such that

$$\left.\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\right|_{\alpha=\frac{s}{v_H}} > \left.\frac{d\gamma^C(\alpha)}{d\alpha}\right|_{\alpha=\frac{s}{v_H}}.$$

Now

$$\begin{aligned}\left.\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\right|_{\alpha=\frac{s}{v_H}} &= \left(\frac{s}{\alpha^2 v_H}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right] + \left(1-\frac{s}{\alpha v_H}\right)\frac{d}{d\alpha}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right]\right)\Bigg|_{\alpha=\frac{s}{v_H}} \\ &= \frac{1}{\alpha}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right],\end{aligned}$$

where the first equality follows from direct calculation (compare with equation (9)) and the second equality follows from evaluating at $\alpha = \frac{s}{v_H}$. Likewise, we have that

$$\begin{aligned}\left.\frac{d\gamma^C(\alpha)}{d\alpha}\right|_{\alpha=\frac{s}{v_H}} &= \left(\frac{s}{\alpha^2 v_H}[1-(1-\alpha)^n] + \left(1-\frac{s}{\alpha v_H}\right)\frac{d}{d\alpha}[1-(1-\alpha)^n]\right)\Bigg|_{\alpha=\frac{s}{v_H}} \\ &= \frac{1}{\alpha}[1-(1-\alpha)^n],\end{aligned}$$

[51] As usual, the critical discount factor $\overline{\delta}$ above which the optimal collusive scheme can be supported via grim-trigger strategies is given by $\overline{\delta} = \frac{\pi^D - \pi^C}{\pi^D - \pi^N}$. For each parameter constellation, π^N is known from Propositions 2 to 4, while π^C is known from the above analysis. It is moreover easy to see that in the high-price equilibrium region ($p^C = v_H - \frac{s}{\alpha}$) and Bertrand-equilibrium region ($p^C = \alpha v_H + (1-\alpha)v_L - s$), the respective optimal deviation from p^C is always to undercut marginally, giving rise to a single optimal deviation profit and thereby a single critical discount factor in either region. However, in the gap equilibrium region and low-price equilibrium region, which share the same π^N, either $p^C = v_H - \frac{s}{\alpha}$ or $p^C = v_L$ can constitute the optimal collusive price level, depending on v_L/v_H. Moreover, it can be shown that in both cases, the optimal deviation can be to undercut marginally *or* to undercut substantially to stop all consumers from searching on. This implies that four different critical discount factors emerge in these regions. In total, one of six critical discount factors is thus the relevant one for supporting the optimal collusive scheme via grim-trigger strategies. Since this is not the main interest of this article, the various critical discount factors are not reported here. Further details are available from the authors upon request.

where the first equality follows from direct calculation (compare with equation (21)) and the second equality follows from evaluating at $\alpha = \frac{s}{v_H}$. Hence, it holds that $\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}} > \frac{d\gamma^C(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}}$ if and only if

$$h_1(\alpha,n) := \frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha} > 1-(1-\alpha)^n := h_2(\alpha,n).$$

Observe next that $h_1(0,n) = \frac{1}{2} > 0 = h_2(0,n)$. Hence, for $\alpha = 0$, it holds that $h_1(\alpha,n) > h_2(\alpha,n)$, such that by continuity of h_1 and h_2 in α, the equality remains true also for slightly positive α, say up to $\alpha_{max}(n) > 0$. When now s is sufficiently small with $s \in [0, \alpha_{max}(n)v_H]$ such that $\frac{s}{v_H} \leq \alpha_{max}(n)$, the inequality $\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}} > \frac{d\gamma^C(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}}$ is thus satisfied. This completes the proof. □

Proof of Proposition 11. As argued in the main text, from the cartel's perspective it can only be optimal that all firms choose the highest possible price which induces search, $p_i^K = v_H - \frac{s}{\alpha}$ for all $i \in \{1,...,n\}$, or that exactly one firm sets the low price v_L while all other $n-1$ firms set the highest possible price $v_H - \frac{s}{\alpha}$ (compare also with Footnote 41). The former implies a cartel profit of $\pi^K = \left(v_H - \frac{s}{\alpha}\right)[1-(1-\alpha)^n]$, the latter a cartel profit of

$$\pi^K = v_L[\alpha+(1-\alpha)^n] + \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)\left[1-(1-\alpha)^{n-1}\right].$$

Comparing these two, it turns out that the former is strictly better if and only if

$$\frac{v_L}{v_H} < \left(1-\frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha+(1-\alpha)^n} =: \gamma^K,$$

as reported in the proposition. That $\gamma^K > \underline{\gamma}$ is trivial to check, whereas $\gamma^K < \gamma^C$ has already been shown in Footnote 42. The final welfare statements are obvious when comparing the different regimes. □

Price-Directed Search and Collusion*

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Abstract

In many (online) markets, consumers can readily observe prices, but need to examine individual products at positive cost in order to assess how well they match their needs. We propose a tractable model of price-directed sequential search in a market where firms compete in prices. Each product meets consumers' basic needs, however they are only fully satisfied with a certain probability. In our setup, four types of pricing equilibria emerge, some of which entail inefficiencies as not all consumers are (always) served. We then lend our model to analyze collusion. We find that for any number of firms, there exists a parameter region in which the payoff-dominant symmetric collusive equilibrium gives rise to a higher expected total social welfare than the repeated one-shot Nash equilibrium. In other regions, welfare is identical under collusion and merely consumer rents are transferred, or both welfare and consumer rents are reduced. An all-inclusive cartel maximizing industry profit increases welfare for an even larger set of parameters, but may also be detrimental to it.

Keywords: Consumer Search, Directed Search, Price Competition, Mixed-Strategy Pricing, Collusion, Cartels

JEL Classification: D43, D83, L13

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1 Introduction

Imagine you are looking for a hotel room to spend a weekend city trip. You open your favorite online platform, specify the city and date, and also select, as far as this is possible on the given platform, all additional criteria that your hotel has to satisfy: at least four stars, inner-city location, free wireless access, etc. You are then presented a list of suitable options, perhaps ordered by price (or at least, with prices prominently displayed), and may click on each individual option to obtain further details (such as pictures and a full description of the hotel's amenities), as well as the ability to book a room.

Of course, it will typically be difficult to assess the attractiveness of any given hotel before inspecting it more closely. For example, only after checking some pictures, a more detailed description and/or customer reviews, you may learn about a hotel's modern style and lively location, which you value (while other consumers may prefer a traditional house in a quiet neighborhood). Yet, your basic needs (e.g., that you can stay at a four-star hotel in the city center) will certainly be satisfied by all showcased alternatives. Moreover, since the presented options appear ex-ante identical (as all satisfy your specified criteria), it will make sense to search through them from lowest to highest price, stopping (booking) when you find something sufficiently nice.[1]

We believe that not only consumers' search for hotel rooms, but many (online) search problems can be described by a similar structure: prices (in a given product category) are readily observable, the competing products all satisfy consumers' basic needs, but consumers need to search through them at positive cost (e.g, as this requires time and effort) to be able to assess how much they like them *beyond* a certain base utility.

In order to capture the spirit of this setting while keeping the analysis tractable, we set up a model of price-directed sequential search in which consumers' *match values* (i.e., how much they like the ex-ante symmetric firms' heterogeneous products) are binary: for each firm's single product, a consumer's valuation can either be low (a partial match) – but

[1] Baye et al. (2009) provide evidence that the number of clicks received by online retailers is indeed highly dependent on their price rank. Examining a large price-comparison site at the time, they find that the lowest-priced retailers for a given product received on average 60% more clicks than higher-priced competitors (even though the authors lack data how this was eventually reflected in final demand). Relatedly, Ellison and Ellison (2009) document that the price transparency provided by a price search engine tended to make demand (for low-quality computer memory modules, a relatively homogeneous good) extremely elastic, even though this was counteracted by obfuscation attempts by some of the examined online retailers.

still, above firms' marginal cost – or high (a full match), following an exogenous two-point distribution. We solve for consumers' optimal search procedure for any combination of prices and exogenous parameters (number of firms, value of partial matches relative to full matches, probability of full matches) and proceed to study firms' equilibrium pricing.

We find that, depending on parameters, one of four types of unique symmetric pricing equilibria emerges. First, if consumers' search costs are large and/or product differentiation is small (in the sense that partial matches provide a similar utility to full matches), firms deterministically price at marginal cost, while consumers search exactly one random firm, buying there no matter whether a full or partial match is found. Marginal-cost pricing occurs because under the described circumstances, no consumer ever searches on after starting at (one of) the lowest-priced firm(s), giving rise to Bertrand-type competition.

If instead the search costs are not too large while product differentiation is sufficiently large, firms can sustain positive profits in equilibrium. Due to undercutting incentives to be searched earlier, firms draw prices randomly from an atomless distribution bounded away from marginal cost, and consumers search orderly from lowest to highest price, only stopping and buying when they find a full match at some firm (but potentially, returning to the lowest-priced firm if no full match is found at any firm). We show that the mixed-strategy equilibrium comes in three subtypes, depending on the degree of product differentiation. For a relatively low differentiation, a "low-price equilibrium" emerges in which all firms always price below the valuation of partially-matched consumers, such that all consumers purchase eventually. In contrast, for a relatively high differentiation, a "high-price equilibrium" results in which all firms always price above the valuation of partially-matched consumers, such that consumers without a full match at any firm drop out deterministically. Finally, for an intermediate differentiation, a "gap equilibrium" occurs in which the firms randomize between pricing below or above the valuation of partially-matched consumers – with a gap just above this valuation – such that consumers without a full match at any firm drop out with positive probability.

In terms of welfare, it is obvious that the latter two equilibria entail inefficiencies, as not all consumers are (always) served, despite all consumers having a valuation above marginal cost (hence, in a second-best scenario in which firms were forced to price below the valuation of partially-matched consumers, welfare would be strictly higher in expectation). The cause of this welfare loss differs from that in existing tractable models of price-directed search such as Ding and Zhang (2018) (see also the detailed literature discussion below), which generally assume an all-or-nothing structure of product matches (either a full match or no match at all).

We also consider the comparative statics of social welfare, firm and industry profits and consumer surplus with respect to the market parameters. For welfare, we find that the comparative statics generally go in the expected direction: welfare increases in the value and probability of full matches, value of partial matches and number of firms, and decreases in consumers' search cost – which we view as strength of our model.[2] As other noteworthy findings, we show that an increase in the number of firms may increase industry profit by expanding the fraction of consumers with at least one full match (and thereby, a high willingness to pay), and that seemingly positive changes for consumers (such as an increase in the value of full matches or a decrease in search costs) may actually harm consumers by dampening competition.

We would like to propose our framework as realistic, yet flexible and tractable model of price-directed search. To showcase its usefulness, we study, as main application, the consequences of collusion and cartelization on market outcomes. Lately, suspicions arose that the increased transparency in online markets has led to firms colluding to keep prices high, such as pointed out by the European Commission[3]:

> "Additionally the price transparency which comes with e-commerce provides possibilities to easily monitor the price setting behaviour of competitors and retailers. Many companies use pricing software which automatically adjusts prices to those of competitors."

Considerable attention has since been given to firms using algorithms which take changes in prices of their competitors into account.[4] The European Commission then conducted inves-

[2]Yet, in the gap-equilibrium region, also counter-intuitive reversed comparative statics may occur locally. This is because firms may strategically respond to changes in some of the parameters by shifting probability mass to high prices (for an increase in the value and probability of full matches) or to low prices (for an increase in search costs), which, by increasing deadweight loss in the former case and decreasing it in the latter, may dominate the positive (negative) direct effects on welfare.

[3]See https://one.oecd.org/document/DAF/COMP/WD(2018)61/en/pdf, p.3., accessed March 14, 2019.

[4]See https://www.fastcompany.com/90311848/yes-retailers-are-colluding-to-inflate-prices-online or https://www.bizcommunity.com/Article/196/394/188110.html. (both accessed March 14, 2019).

tigations on pricing algorithms employed by firms in e-commerce and to which extent they qualify as anti-competitive practices.[5] They found that especially in a horizontal context, the pricing algorithms facilitate explicit *and implicit, tacit* collusive agreements.

While other recent contributions such as Petrikaitė (2016) have also investigated collusion in markets characterized by consumer search, we are, to the best of our knowledge, the first to combine an analysis of price-directed search and collusion. We show that the payoff-dominant symmetric collusive equilibrium (which can e.g. be supported by grim-trigger strategies if firms are sufficiently patient) has firms coordinating on one of three price levels, depending on parameters. For either very low or high product differentiation, the optimal collusive price level coincides with the highest price that still keeps consumers in the market, such that consumer rents are clearly transferred to firms. Moreover, under moderately high differentiation, also social welfare may be reduced, as part of the consumers drop out deterministically in the collusive equilibrium, while they may be served with positive probability in the corresponding (one-shot) Nash equilibrium of the baseline game.

However, interestingly, for intermediate levels of product differentiation, the profit-maximizing collusive price level lies at consumers' valuation for partial matches, such that all consumers are served deterministically under collusion, and welfare is at the second best. When comparing this to the one-shot Nash outcome under the same parameters, it turns out that this type of collusive equilibrium may actually *increase* welfare (and at the very least does not decrease it). Welfare increases when without collusion, the gap equilibrium would be played, in which case firm coordination eliminates the probabilistic deadweight loss which would arise under unconstrained competition. We establish that for any number of firms, there is indeed a parameter region where this occurs, namely when both the value of partial matches relative to full matches is not too high, and the probability of full matches is intermediate.

Finally, we examine the market outcome under an all-inclusive cartel (alternatively, if all firms merge to a multi-product monopolist). Once again, we find that one of three different price configurations is optimal, two of which – again under very low differentiation and high differentiation – have all firms choosing the maximum price which keeps consumers in the market. However, it can no longer be profit-maximizing to collectively price at the valuation

[5] See `https://one.oecd.org/document/DAF/COMP/WD(2017)12/en/pdf`, accessed March 14, 2019.

of partially-matched consumers: the same demand (i.e., all consumers in the market), but a higher profit, can now be achieved when just one firm sets this low price. The low-priced product then serves as “compromise option” for those consumers who don’t have a full match at any firm, while all other products are priced maximally and sold to fully-matched consumers. Whenever this is the cartel solution, welfare is at the second best, and we show that a welfare improvement through cartelization occurs for an even larger set of parameters than under symmetric collusion.

Related Literature. Our paper joins an extensive literature on costly consumer search, studying the effects of frictions and incomplete information about product characteristics and/or prices on market outcomes. For comprehensive literature reviews see Anderson and Renault (2018) and Baye et al. (2006), or, for the case of digital markets, Moraga-González (2018).

In early work which relates to our model, such as the seminal papers by Wolinsky (1986), Stahl (1989) and Anderson and Renault (1999), prices are unobservable and consumer search is random. Departing from models of random search, there have been efforts to describe environments in which consumers search firms according to some order. The first papers in this vein focused on predetermined orders, arising naturally e.g. when thinking about geographical distance (see Arbatskaya (2007) for homogeneous products, Armstrong et al. (2009) for differentiated products with a “prominent” firm[6], or Zhou (2011) for a general analysis with differentiated products). In Athey and Ellison (2011) and Chen and He (2011), firms bid for positions along consumers’ search path. However, in these models, prices do not influence the order of search. Armstrong (2017) outlines a setting in which the order of search is chosen endogenously by consumers forming *expectations* about prices and firms acting according to their beliefs in equilibrium.

One of the first attempts to model *observable* prices as important strategic variables for directing search can be found in Armstrong and Zhou (2011, Section 2), where firms advertise the price of their differentiated product on a price-comparison website. Consumers’ optimal search path is then guided by those advertised prices. To keep the model tractable, Armstrong and Zhou introduce a specific (Hotelling duopoly) structure in which consumers’

[6] That is, one firm is exogenously searched first by all consumers, while the remaining firms are searched in random order.

match values are perfectly negatively correlated.[7] A main finding is that the competition among firms to receive a larger market share by being sampled first drives down retail prices, relative to a benchmark model without price advertising, and that this effect is stronger when search frictions increase.

Tractability is generally a major issue when it comes to solving models of price-directed search. For example, even a duopoly version of the standard differentiated-products framework by Wolinsky (1986) with independently distributed match values becomes intractable with observable prices, as the resulting mixed-strategy equilibrium is extremely hard to characterize. Haan et al. (2018) and Choi et al. (2018) circumvent this problem by incorporating sufficiently strong ex-ante differentiation into Wolinsky's framework with observable prices.[8] This restores existence of a pure-strategy equilibrium that can be characterized. By considering a two-point distribution of match values, we obtain tractability without introducing any exogenous ex-ante differentiation.

In terms of its underlying model, our paper is most closely related to Ding and Zhang (2018), which also studies price-directed search in a market with differentiated products and ex-ante homogeneous firms. There are two major differences. First, we do not include informed consumers who costlessly observe all match values, which is however crucial to generate most interesting results in Ding and Zhang.[9] Second, and most important, we allow for product differentiation to be more nuanced. While in Ding and Zhang consumers either fully value a product or not at all, in our setting they may have a positive willingness to pay for all products. As a result, consumers may optimally *return* to purchase from a previously sampled firm, which affects competition and has important consequences for market outcomes that are otherwise not captured.[10]

[7]More concretely, upon inspecting the lower-priced product first, consumers learn its match value and can then perfectly deduce the match value offered by the other firm.

[8]See also Shen (2015) for related analysis in a Hotelling context.

[9]For example, they also find circumstances where a "gap equilibrium" occurs, but this is only the case if the fraction of informed consumers is sufficiently large. Especially for markets where many consumers are first-time buyers, such as our motivating example of lodging services, consumers who know their match values in advance (or can search them for free) arguably constitute a small minority. For simplicity and to highlight a different channel, we set their number to zero in our model. For a sufficiently small fraction of informed consumers, our qualitative results would remain similar.

[10]As a third distinction to Ding and Zhang (2018), consumers' first search is not costless in our model. While we anyway consider a positive search cost for every sampled product to be more realistic, the equilibrium characterization of our baseline model would remain virtually unchanged with costless first search (just the parameter region where the market is inactive would vanish). Of course, our comparative-statics analysis of various welfare measures, as well as the specification of the optimal collusive schemes, would have to be adapted.

Our analysis of firm coordination is related to a vast theoretical literature on collusion.[11] However, applications to markets with search frictions have been limited, and mainly focused on the *sustainability* of collusion rather than on its welfare effects. Petrikaitė (2016) contrasts models of non-directed consumer search with differentiated and homogeneous products to investigate how cartel stability is affected by search costs. She finds that increased search costs facilitate collusion if products are differentiated, while the opposite is true if products are homogeneous. In the homogeneous-products duopoly model with imperfect monitoring studied by Campbell et al. (2005), increased search costs also make collusion harder to sustain. Schultz (2005) considers a Hotelling framework in which only a fraction of consumers is aware of both firms' prices.[12] He finds that an increase in market transparency through a higher fraction of informed consumers decreases the scope for collusion. In a model of non-sequential search for homogeneous products, Nilsson (1999) finds the opposite, namely that an increase in market transparency through lower search costs may promote collusion. Overall, the specific market environment seems decisive.[13]

In contrast to the aforementioned papers, our model features observable prices *for all* market participants. It is thus easy for firms to detect and punish deviations from the (tacit or explicit) collusive agreements. Further, prices actually *direct* search. Compared to models of random search, this has a notable effect on firms' incentives to collude, as they are able to directly influence consumers' search order by deviating from a collusive agreement. A major novel finding in our paper is that firms avoiding competition can have no *or even a positive* effect on total welfare.

The remainder of this article is structured as follows. Section 2 introduces the model setup, while in Section 3, we solve the baseline model and discuss its welfare implications and comparative statics. In Section 4, we reformulate our baseline model as infinitely repeated game and analyze the payoff-dominant symmetric equilibria, as well as the cartel outcome. We also compare welfare to the baseline model. Section 5 concludes. Several technical proofs are relegated to the Appendix.

[11] See Feuerstein (2005) for a survey.

[12] See also Schultz (2017), which extends the author's earlier model to allow for limited price observability on the supply side as well.

[13] The question whether or not collusion is facilitated in markets with costly consumer search has also been addressed empirically. Moraga-González et al. (2015) investigate the market for car dealerships by employing models of random search. While they do not find strong evidence for collusion in the examined markets, they show that lowering consumers' search costs can indeed lead to higher prices, as is consistent with our baseline results. Nishida and Remer (2018) also find detrimental effects of reduced search costs studying the retail gasoline market.

2 Model Setup

Consider the following market. There are $n \geq 2$ risk-neutral firms $i = 1, \dots, n$ that compete in prices p_i. Each firm offers a single differentiated product. Firms' constant marginal costs of production are normalized to 0.

There is a unit mass of risk-neutral consumers with unit demand and an outside-option value that is normalized to zero. Each consumer freely observes the prices of all products. However, consumers are initially unaware whether any given product will be a full or partial match for them. Precisely, product i perfectly suits a consumer's needs (the product is "a full match") with probability $\alpha \in (0,1)$. In case of a full match, consumers' willingness to pay is given by $v_i = v_H > 0$. With complementary probability $1-\alpha$, product i is only "a partial match", for which consumers' willingness to pay is given by $v_i = v_L \in [0, v_H]$. We assume that the match values v_i are identically and independently distributed across each consumer-firm pair, and that the firms are unable to identify which product(s) will be a match for any individual consumer, ruling out price discrimination.

In order to find out their match values, consumers have to incur a search cost $s \geq 0$ per product that they sample. It is assumed that they cannot purchase any product before searching it first. Consumers engage in optimal sequential search with free recall and maximize their expected consumption utility, where consumption utility is given by

$$u_i := v_i - p_i - ks, \qquad \text{with } v_i \in \{v_L, v_H\} \tag{1}$$

when buying product i (which can either be a full or partial match) after having searched $k \in \{1, \dots, n\}$ products, and $u_0 = -ks$ when taking their outside option after having searched $k \in \{0, \dots, n\}$ products. All market parameters are common knowledge.

The timing of the game is as follows. First, firms simultaneously set prices p_i. Second, consumers observe these prices, and engage in optimal sequential search. Third, payoffs realize.

In order to make the problem interesting, we finally assume that $\alpha v_H + (1-\alpha)v_L - s \geq 0$. Otherwise, the market collapses, as no firm could offer a non-negative expected surplus to consumers even when setting $p_i = 0$.

3 Equilibrium Analysis

Optimal Search. Since, apart from their prices, firms' products appear ex-ante identical, consumers will clearly find it optimal to search firms in ascending order of their prices.[14] Without loss of generality, we index firms such that $p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq p_n$. Given a consumer started at firm 1 and found a full match, the consumer optimally purchases, since there can be no gain from searching on. However, if only a partial match is found at firm 1, the consumer might want to continue to search firm 2, and so on. Consumers' optimal search behavior now crucially depends on whether $p_1 > v_L$ or $p_1 \leq v_L$, as only in the latter case, consumers may want to return to purchase at firm 1 in the course of their search process. The following lemma fully characterizes consumers' optimal search behavior.

Lemma 1. *Optimal Search:*

- *If $p_1 > v_L$, search, in increasing order of prices, all firms $i = 1, \ldots, n$ for which $p_i \leq v_H - \frac{s}{\alpha}$. Purchase immediately if a full match is found, and search on if not. If no full match is found at any suitable firm, take the outside option.*
- *If $p_1 \leq v_L$, start search at firm 1 if $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$, and otherwise take the outside option. Given firm 1 is searched and a full match is found, purchase there immediately. If not, search, in increasing order of prices, all firms $i = 2, \ldots, n$ for which $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$. Purchase immediately if a full match is found, and search on if not. If no full match is found at any suitable firm, purchase at firm 1.*

Proof. The first part is straightforward: Given that all prices exceed v_L, consumers will only buy from a firm if it provides a full match, and as long as no full match has been found, consumers hold a utility of zero. Hence, provided that no full match has been found yet, the expected one-shot gains from searching any firm i are given by $\alpha(v_H - p_i) - s$, which is non-negative if and only if $p_i \leq v_H - \frac{s}{\alpha}$. It is therefore optimal to search, in increasing order of prices, all firms for which this holds, and purchase immediately if a full match is found. If no full match is found at any firm which satisfies $p_i \leq v_H - \frac{s}{\alpha}$, a consumer optimally takes the outside option.

[14] In case of ties, consumers are assumed to randomize with equal probability between firms, which is however inconsequential for our results. We moreover assume that whenever a consumer is indifferent between purchasing directly and searching on, the consumer searches on, and whenever a consumer is indifferent between buying and not buying after their search process, the consumer buys.

If, on the other hand, $p_1 \leq v_L$, the expected one-shot gains of searching any firm are clearly largest for firm 1 and if no other firm has been searched yet. Hence, a consumer should only start to search (at firm 1) if the expected one-shot gains of doing so, $\alpha(v_H - p_1) + (1-\alpha)(v_L - p_1) - s$, are non-negative. This transforms to $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$. If this holds and it is therefore optimal to search firm 1, consumers should clearly purchase there immediately if a full match is found. If a partial match is found, a consumer holds a purchase option of value $v_L - p_1 \geq 0$, which remains true as long as only partial matches have been found at every searched firm. Hence, provided that only partial matches have been found so far, the expected one-shot gains from searching any firm $i = 2, \ldots, n$ are given by $\alpha((v_H - p_i) - (v_L - p_1)) - s$. This is non-negative for all firms i which satisfy $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$. It is therefore optimal to search these firms in increasing order of their prices and purchase immediately if a full match is found. If no full match is found at any firm which satisfies $p_i \leq p_1 + (v_H - v_L - \frac{s}{\alpha})$, a consumer optimally returns to purchase from firm 1. $\square$

Preliminary Equilibrium Results. Having characterized consumers' optimal search behavior, one may first note that for $v_H - v_L - \frac{s}{\alpha} \leq 0$, the binding condition for consumers to start searching is $p_1 \leq \alpha v_H + (1-\alpha)v_L - s$ $(\leq v_L)$; moreover, consumers will never search firms that are not among the lowest-priced. The reason is that in the considered parameter range, after obtaining a partial match at (one of) the lowest-priced firm(s), the expected gains from searching are too low for any higher-priced firms. Intuitively, this is true because the condition $v_H - v_L - \frac{s}{\alpha} \leq 0$ holds if either the probability of finding a full match is very low relative to the search cost ($\alpha \leq \frac{s}{v_H}$), or if this not the case, but partial matches provide a too similar utility to full matches, given the probability of finding a full match and the search cost ($\alpha > \frac{s}{v_H}$, but $\frac{v_L}{v_H} \geq 1 - \frac{s}{\alpha v_H}$). Then, the property that consumers will only search firms which are among the lowest-priced immediately implies the following.

Proposition 1. *Suppose that $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \bar{\gamma} := 1 - \frac{s}{\alpha v_H}$. Then in the unique symmetric equilibrium each firm chooses $p^* = 0$ and earns zero profit. On the equilibrium path, each consumer searches exactly one random firm and buys there immediately, independent of whether a full or partial match is found.*[15]

[15] In the borderline case where $\frac{v_L}{v_H} = \bar{\gamma}$, given that $p_i = 0$ for all firms, consumers are actually indifferent between buying immediately after obtaining a partial match or searching on. This is however inconsequential.

Proof. See the argument above. Given $p^* = 0$, consumers indeed find it optimal to search one random firm due to the parameter assumption of $\alpha v_H + (1-\alpha)v_L - s \geq 0$. $\square$

We will subsequently refer to the parameter region where Proposition 1 holds as "Bertrand region", since intense price competition drives firms to price at marginal cost. As we show next, the market outcome is decisively different for all other parameter combinations.

Lemma 2. *If $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \bar{\gamma}$, there exists no symmetric pure-strategy equilibrium. In a symmetric mixed-strategy equilibrium, firms make positive expected profits and draw prices from an atomless CDF bounded away from zero.*

Proof. A symmetric pure strategy-equilibrium at any positive price level can never exist because either firm would have an incentive to marginally undercut to be searched first by all consumers, rather than just by $1/n$ of the consumers. However, unlike the case where $v_H - v_L - \frac{s}{\alpha} \leq 0$, it is also no equilibrium that every firm prices at marginal cost (i.e., zero). This is because, for $v_H - v_L - \frac{s}{\alpha} > 0$, when all rival firms price at zero, setting a price in the non-empty range $(0, v_H - v_L - \frac{s}{\alpha}]$ guarantees a firm to be searched (by those consumers who did not find a full match at any rival firm, compare with Lemma 1) and make a positive profit. Hence, any symmetric equilibrium must be in mixed strategies. The respective equilibrium pricing CDF must be bounded away from zero because firms can guarantee a positive profit. It must be atomless because otherwise, transferring probability mass from the atom(s) to prices marginally below would pay because this avoids ties. $\square$

Preview of Mixed-Strategy Equilibria. It turns out that the symmetric mixed-strategy equilibrium for the case that $v_H - v_L - \frac{s}{\alpha} > 0$ comes in three qualitatively different subtypes, depending on the degree of product differentiation (which is inversely related to v_L/v_H) in combination with the other market parameters.

In particular, as mentioned in the Introduction, either a "high-price equilibrium" (high differentiation, with $\frac{v_L}{v_H} \leq \underline{\gamma}$), a "low-price equilibrium" (relatively low differentiation, with $\frac{v_L}{v_H} \in [\tilde{\gamma}, \bar{\gamma})$), or a "gap equilibrium" (intermediate differentiation, with $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$) emerges as the unique equilibrium. In the next three subsections, we fully characterize these equilibria in turn. Figure 1 previews the various equilibrium regions in $(\alpha, \frac{v_L}{v_H})$-space for a specific combination of search costs (relative to v_H) and number of firms. Note that in region X in the bottom-left corner, our parameter assumption of $\alpha v_H + (1-\alpha)v_L - s \geq 0$ is violated, such that the market is inactive in this region.

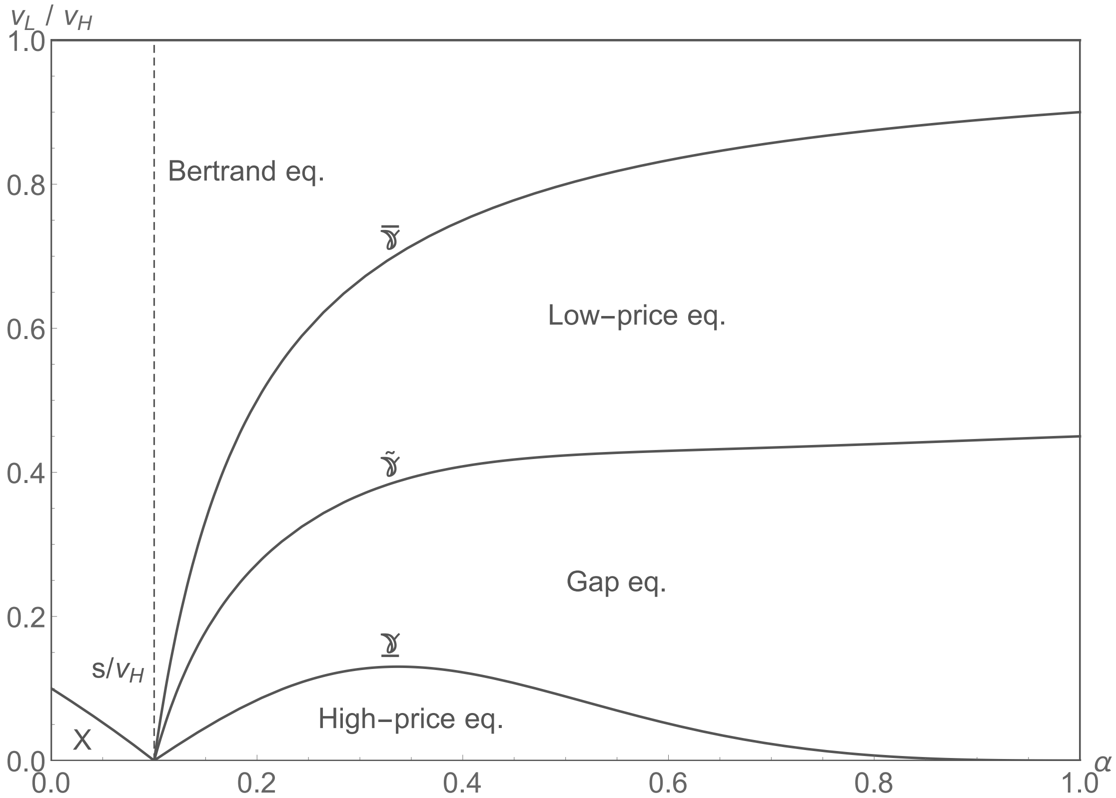

Figure 1: Depiction of equilibrium regions for $\frac{s}{v_H} = 0.1$ and $n = 4$.

3.1 High-Price Equilibrium

As we show, if product differentiation is relatively high, a "high-price equilibrium" emerges. In such an equilibrium, all firms always price strictly above v_L, such that only consumers with a full match may ever buy at any given firm. Moreover, a firm cannot attract any "returning" demand: consumers either buy immediately after having searched some firm, which we refer to as "fresh" demand, or never return. We now construct such a candidate equilibrium.

Since a symmetric equilibrium price distribution must be atomless (see Lemma 2), a firm pricing at the respective upper support bound $\overline{p}_H$ will only be searched by consumers who did not have a full match at any (earlier sampled) rival firm. But then it follows immediately that $\overline{p}_H$ must be equal to the highest price that may ever be searched by consumers, namely

$\overline{p}_H = v_H - \frac{s}{\alpha}$ (compare with Lemma 1). This is because if $\overline{p}_H$ was smaller, a firm choosing $\overline{p}_H$ could profitably deviate upward to the price $v_H - \frac{s}{\alpha}$ instead, for which it would not lose any demand. On the other hand, if $\overline{p}_H$ was larger, a firm setting this price would never be searched.

Having pinned down the equilibrium upper support bound, firms' equilibrium expected profit easily follows: it is given by the highest equilibrium price times the number of consumers a firm can serve at it (the mass of consumers who have no full match at firms 1 to $n-1$, but a full match at firm n), such that $\pi_H^* = \overline{p}_H(1-\alpha)^{n-1}\alpha$. In turn, also firms' equilibrium lower support bound can easily be derived: when a firm chooses the lowest equilibrium price $\underline{p}_H(> v_L)$, it will be searched first by all consumers, and a fraction α of those (with a full match) will purchase there. Hence, $\underline{p}_H$ simply solves $\underline{p}_H\alpha = \pi_H^*$.

The equilibrium CDF $F_H(p)$ can then be found as follows. A firm pricing at some price p in the support of F_H gets an expected profit of

$$\pi_i(p) = p(1-\alpha F_H(p))^{n-1}\alpha,$$

which has to be equal to π_H^* everywhere in the support – solving $\pi_i(p) = \pi_H^*$ for $F_H(p)$ then gives the equilibrium CDF. To understand the above profit expression, consider firm i's probability to sell to any given consumer. Clearly, a consumer will only search firm i if there is not a single rival firm with a lower price that also provides a full match to the consumer. The probability that a single rival firm does not have a lower price *and* provides a full match is given by $1-\alpha F_H(p)$. The probability that none of the $n-1$ rival firms does so is then given by $(1-\alpha F_H(p))^{n-1}$. Hence, with the latter probability, any given consumer searches firm i. This consumer will then buy at firm i if it provides a full match to the consumer, for a total purchase probability of $(1-\alpha F_H(p))^{n-1}\alpha$.

Finally, to see when this equilibrium exists, note that as long as it is well-defined such that $\underline{p}_H > v_L$, pricing in the range $(v_L, \underline{p}_H)$ cannot be optimal, as this does not increase a firm's expected demand. However, it can potentially be optimal to price at v_L or below. Pricing at v_L or below has the advantage that a firm also attracts "returning" demand: those consumers who do not have a full match at the deviating firm will still return if they also have no full match at any other (higher-priced) rival firm. While it may seem intuitive that deviating to a strictly lower price than v_L cannot be better than pricing at v_L, this is actually not immediately obvious, as under the high-price candidate equilibrium, deviation prices strictly below v_L imply a positive probability that consumers will return to purchase from the deviating firm (providing a partial match) without searching all rival firms. In the proof of the subsequent proposition, we show however that the optimal deviation price below $\underline{p}_H$ is indeed always given by v_L, for a maximal deviation profit of $\pi_i^{dev^*} = v_L[\alpha + (1-\alpha)^n]$. This is not higher than the candidate equilibrium profit if and only if $\frac{v_L}{v_H} \leq \underline{\gamma}$, where $\underline{\gamma}$ is defined below. Proposition 2 summarizes the above findings.

Proposition 2. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} \in [0, \underline{\gamma}]$, *where*

$$\underline{\gamma} := \frac{\left(1 - \frac{s}{\alpha v_H}\right)(1-\alpha)^{n-1}\alpha}{(1-\alpha)^n + \alpha}. \tag{2}$$

Then there exists a unique symmetric equilibrium in which each firm samples prices continuously from the interval $[\underline{p}_H, \overline{p}_H]$ *following the atomless CDF*

$$F_H(p) := \frac{1}{\alpha}\left[1 - (1-\alpha)\left(\frac{v_H - \frac{s}{\alpha}}{p}\right)^{\frac{1}{n-1}}\right], \tag{3}$$

with

$$\underline{p}_H := \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1} > v_L \tag{4}$$

and

$$\overline{p}_H := v_H - \frac{s}{\alpha}. \tag{5}$$

Each firm makes an expected profit of

$$\pi_H^* := \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1}\alpha. \tag{6}$$

On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and takes the outside option if no full match is found at any firm.

Proof. See Appendix A. □

Note that various versions of the above pricing equilibrium have appeared before in the literature, where it was generally assumed that $v_L = 0$. Setting $v_L = 0$, the equilibrium in Proposition 2 applies whenever $\alpha > \frac{s}{v_H}$. It is then easy to see that we nest the model of price-directed search by Ding and Zhang (2018) for the case in which there are no informed consumers ($\mu = 0$ in their notation).[16] Letting moreover $s = 0$ and $n = 2$, we nest a duopoly version of Varian (1980) with inelastic demand up to a maximum valuation of v_H (with a fraction $\lambda = \frac{\alpha^2}{1-(1-\alpha)^2} = \frac{\alpha}{2-\alpha}$ of fully-informed "shoppers"). For $s = 0$ and arbitrary $n \geq 2$, our setup is also identical to the second stage of Ireland (1993) when his "information shares" s_i (i.e., the share of consumers who know about the existence of firm i) satisfy $s_i = \alpha$ for all $i = 1, \ldots, n$ (and $v_H = 1$ to match his normalization).[17]

3.2 Low-Price Equilibrium

As we show next, when product differentiation is relatively low, a "low-price equilibrium" results. In this equilibrium, all firms always sample prices below v_L, such that all consumers buy at some firm. It turns out that the equilibrium can then be pinned down by two conditions. The first is that, perhaps surprisingly, the equilibrium lower pricing support bound $\underline{p}_L$ and the equilibrium upper pricing support bound $\overline{p}_L$ lie just so far apart that when having a partial match at the lowest possible price $\underline{p}_L$, a consumer would exactly be indifferent between purchasing there, or searching a firm with the highest possible price $\overline{p}_L$. This implies that on the equilibrium path each consumer keeps searching deterministically until a full match is found, and only returns to the lowest-priced firm in case no full match is found at any firm. Comparing with consumers' optimal search rule for the case where $p_1 \leq v_L$ (see Lemma 1), the relevant condition for this is that

$$\alpha((v_H - \overline{p}_L) - (v_L - \underline{p}_L)) = s. \tag{7}$$

[16]To be precise, consider Ding and Zhang (2018, Proposition 2) for $\mu = 0$, and let $V = v_H$, $\theta = \alpha$ and $N = n$ to match our notation. Then $r = v_H - \frac{s}{\alpha}$ (compare with their equation (2)), and their threshold value s_1' equals αv_H such that part (i) of their Proposition 2 applies. It is then immediate that their equilibrium CDF $R(p)$ coincides with our equilibrium CDF $F_H(p)$ in the high-price equilibrium (and of course, also the equilibrium expected profits are identical).

[17]For $n = 2$ and $s_1 = s_2 = \alpha$, it is straightforward to see that Ireland's second-stage solution coincides with ours (compare with (Ireland, 1993, p.66)). For $n > 2$, this should also be the case, but due to his focus on asymmetric information shares, the comparison of equilibria is less obvious.

Building on this, the second condition is straightforward: A firm's expected profit when pricing at $\underline{p}_L$ is then given by $\pi_i(\underline{p}_L) = \underline{p}_L(\alpha + (1-\alpha)^n)$, as it will have the lowest price, and therefore be sampled first by all consumers with certainty (for a "fresh" demand of α and a "returning" demand of $(1-\alpha)^n$). This must be equal to the firm's expected profit when pricing at $\overline{p}_L$, which is then, since the firm will have the highest price deterministically, given by $\pi_i(\overline{p}_L) = \overline{p}_L(1-\alpha)^{n-1}\alpha$. Hence, it is required that

$$\underline{p}_L(\alpha + (1-\alpha)^n) = \overline{p}_L(1-\alpha)^{n-1}\alpha. \tag{8}$$

Simultaneously solving equations (7) and (8) gives the candidate $\underline{p}_L$, $\overline{p}_L$ and equilibrium profit π_L^*. The candidate equilibrium is well-behaved (in the sense that $\underline{p}_L \geq 0$ and $\overline{p}_L \leq v_L$) under the condition on $\frac{v_L}{v_H}$ that will be provided in the subsequent proposition. In the proof of the proposition, we also show that deviation prices outside the equilibrium support are not profitable: while pricing below $\underline{p}_L$ still increases a firm's expected demand as this leads to a positive probability that consumers with only a partial match at that firm will return before sampling all rival firms, the respective loss of margin more than outweighs the positive effect on demand. Similarly, while pricing above $\overline{p}_L$ still generates a positive expected demand, this demand decreases sufficiently fast such as to render such deviations unprofitable. This is because by pricing above $\overline{p}_L$, a firm risks that even consumers who did not have a full match at any rival firm will not search it, as they may rather prefer to return to the lowest-priced rival (with only a partial match).

Unfortunately, for the general n-firm case, the equilibrium CDF $F_L(p)$ cannot be obtained in closed form.[18] On the other hand, it is implicitly defined by a straightforward condition, and it is easy to see that it is well-behaved (strictly increasing) for any number of firms. In particular, note that when a firm chooses any price $p \leq v_L$ in the support of F_L, its expected profit is given by

$$\pi_i(p) = p\left[\alpha(1-\alpha F_L(p))^{n-1} + (1-F_L(p))^{n-1}(1-\alpha)^n\right],$$

which, for the equilibrium CDF $F_L(p)$, needs to be equal to π_L^*. To understand the above profit equation, notice that the first term of expected demand at price p in the squared bracket

[18]For $n = 2$ ($n = 3$), it can however be obtained as the solution to a simple linear (quadratic) equation.

follows the logic outlined in the description of the high-price equilibrium. This corresponds to a firm's "fresh" demand: those consumers who search firm i (because they have no full match at any lower-priced rival firm, if any) and have a full match at firm i. The second term of expected demand in the squared bracket is "returning" demand. Indeed, with a probability of $(1-F_L(p))^{n-1}$, all rival firms choose a higher price than p. Then, the considered firm i will attract the mass $(1-\alpha)^n$ of consumers who did not find a full match at any firm (including firm i), who ultimately return to firm i.[19] Proposition 3 summarizes our findings.

Proposition 3. *Suppose that $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in [\tilde{\gamma}, \bar{\gamma})$, where*

$$\tilde{\gamma} := \frac{\left(1-\frac{s}{\alpha v_H}\right)\left[\alpha+(1-\alpha)^n\right]}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha} \tag{9}$$

$$\bar{\gamma} := 1-\frac{s}{\alpha v_H}. \tag{10}$$

Then there exists a unique symmetric equilibrium in which each firm samples prices continuously from the interval $[\underline{p}_L, \overline{p}_L]$ following an atomless CDF $F_L(p)$ that is defined implicitly by

$$p\left[\alpha(1-\alpha F_L(p))^{n-1}+(1-F_L(p))^{n-1}(1-\alpha)^n\right] = \pi_L^*, \tag{11}$$

where

$$\pi_L^* := \frac{(1-\alpha)^{n-1}\left[(1-\alpha)^n+\alpha\right]\left[\alpha(v_H-v_L)-s\right]}{(1-\alpha)^{n-1}(1-2\alpha)+\alpha} \tag{12}$$

denotes each firm's equilibrium expected profit,

$$\underline{p}_L := \frac{(1-\alpha)^{n-1}\left[\alpha(v_H-v_L)-s\right]}{(1-\alpha)^{n-1}(1-2\alpha)+\alpha} \tag{13}$$

and

$$\overline{p}_L := \frac{\left[(1-\alpha)^n+\alpha\right]\left[\alpha(v_H-v_L)-s\right]}{\alpha\left[(1-\alpha)^{n-1}(1-2\alpha)+\alpha\right]} \leq v_L. \tag{14}$$

[19] Note that, since the probability that exactly $k \in \{0,\dots,n-1\}$ of the $n-1$ rival firms price lower than p is given by $\binom{n-1}{k}F_L(p)^k(1-F_L(p))^{n-1-k}$, the "fresh" demand can alternatively be found by computing

$$\sum_{k=0}^{n-1}\binom{n-1}{k}F_L(p)^k(1-F_L(p))^{n-1-k}(1-\alpha)^k\alpha = \alpha\left[(1-\alpha)F_L(p)+(1-F_L(p))\right]^{n-1} = \alpha(1-\alpha F_L(p))^{n-1},$$

where the first equality follows from the binomial theorem. In the event that $k=0$, which happens with probability $(1-F_L(p))^{n-1}$, next to the corresponding "fresh" demand of α, a firm also receives the "returning" demand $(1-\alpha)^n$.

On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and returns to purchase at the lowest-priced firm if no full match is found at any firm.

Proof. See Appendix A. □

3.3 Gap Equilibrium

The arguably most interesting equilibrium arises if v_L takes on intermediate values ($\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$). Then, the high-price equilibrium as characterized in Proposition 2 does not exist because a deviation to v_L is profitable (since $\frac{v_L}{v_H} > \underline{\gamma}$), while the low-price equilibrium as characterized in Proposition 3 does not exist because it would hold that $\overline{p}_L > v_L$ (since $\frac{v_L}{v_H} < \tilde{\gamma}$).

Instead, in the resulting "gap equilibrium" firms draw prices from two disconnected intervals: they either choose low prices in some range $[\underline{p}_M, v_L]$ up to v_L, or they choose high prices in some range $[\underline{p}'_M, \overline{p}_M]$ strictly above v_L. Intuitively, firms' pricing support has a gap right above v_L because firms' demand drops discretely at v_L. This is because when pricing at v_L, there is some positive probability that a firm has the lowest price *and* attracts the mass $(1-\alpha)^n$ of "returning" demand with no full match at any firm, while when pricing at $v_L + \varepsilon$, no consumers would return even if the firm had the lowest price in the market.

Also the gap equilibrium satisfies the two conditions that hold for the low-price equilibrium: the range of firm's equilibrium pricing, $\overline{p}_M - \underline{p}_M$, is again exactly[20] such that on the equilibrium path, every consumer keeps searching until a full match is found (compare with equation (7)), while naturally, pricing at the lowest equilibrium price $\underline{p}_M$ must yield the same expected profit as pricing at the highest equilibrium price $\overline{p}_M$ (compare with equation (8)). It therefore turns out that the equilibrium lower support bound, upper support bound and profit take on the same functional form as in the low-price equilibrium (but now with $\overline{p}_M = \overline{p}_L > v_L$). At the bottom of the upper pricing interval $\underline{p}'_M \in (v_L, \overline{p}_M)$, the expected profit (without any "returning" demand, but with equal "fresh" demand as when pricing at v_L) must then be equal to the expected profit when pricing at v_L (with "returning" demand).

[20] With this, we mean that like in the low-price equilibrium, $\underline{p}_M$ and $\overline{p}_M$ lie just so far apart that when having a partial match at the lowest possible price $\underline{p}_M$, a consumer would exactly be indifferent between purchasing there, or searching a firm with the highest possible price $\overline{p}_M$.

The equilibrium CDFs for the two intervals are finally obtained by essentially the same profit-indifference conditions as in the low-price equilibrium (lower interval) and high-price equilibrium (upper interval), respectively. The only difference is that in the upper interval, the expected profit at any equilibrium price needs to be equal to the equilibrium profit $\pi_M^* = \pi_L^*$, rather than π_H^* in the high-price equilibrium. We further show that in the relevant parameter range, the gap equilibrium is well-behaved, and that there are no profitable deviation prices outside the equilibrium support. Proposition 4 summarizes our findings. A graphical illustration of the equilibrium CDF in the gap equilibrium is provided in Figure 2.

Proposition 4. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$. *Then there exists a unique symmetric equilibrium in which each firm samples prices from two disconnected intervals* $[\underline{p}_M, v_L] \cup [\underline{p}'_M, \overline{p}_M]$, *with* $\underline{p}'_M > v_L$. *In the lower interval, firms draw prices from the atomless CDF* $F_{M_1}(p) := F_L(p)$ *as defined in Proposition 3, while in the upper interval, firms draw prices from the atomless CDF*

$$F_{M_2}(p) := \frac{1}{\alpha}\left[1 - \left(\frac{\pi_L^*}{\alpha p}\right)^{\frac{1}{n-1}}\right], \tag{15}$$

with π_L^*, $\underline{p}_M := \underline{p}_L$ *and* $\overline{p}_M := \overline{p}_L$ *as defined in Proposition 3 and*

$$\underline{p}'_M := \frac{\pi_L^*}{\alpha(1-\alpha\kappa)^{n-1}}, \tag{16}$$

where $\kappa := F_{M_1}(v_L)$ *is implicitly defined by*

$$v_L\big[\alpha(1-\alpha\kappa)^{n-1} + (1-\kappa)^{n-1}(1-\alpha)^n\big] - \pi_L^* = 0, \tag{17}$$

and $F_{M_2}(\underline{p}'_M) = F_{M_1}(v_L) = \kappa$. *Each firm makes an expected profit of* $\pi_M^* := \pi_L^*$. *On the equilibrium path, each consumer keeps searching (in increasing order of prices) until a full match is found, and returns to purchase at the lowest-priced firm if* $p_1 \leq v_L$ *and no full match is found at any firm.*

Proof. See Appendix A. □

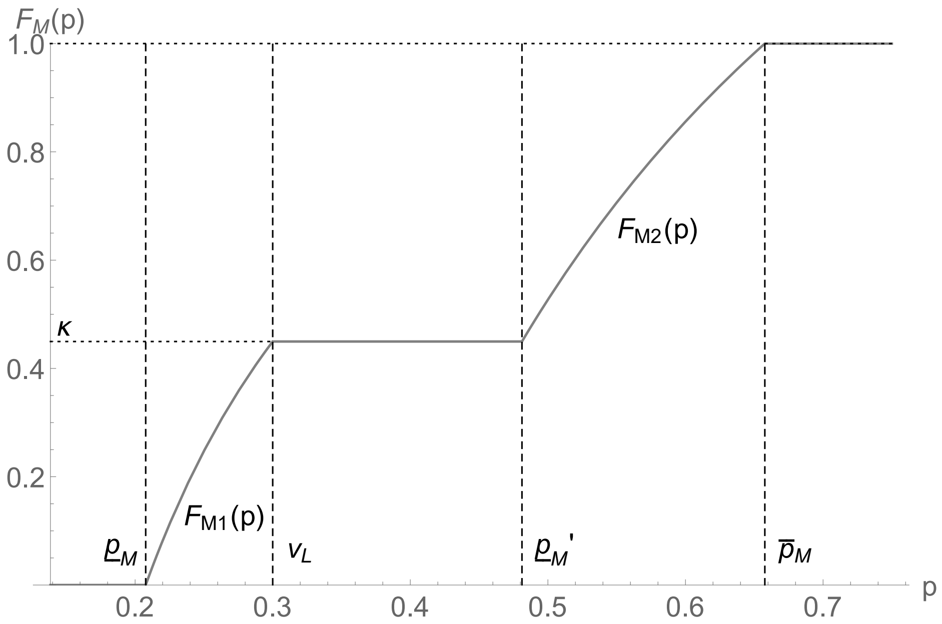

Figure 2: Example equilibrium CDF in the gap equilibrium. The parameters used are $v_H = 1$, $v_L = 0.3$, $s = 0.1$, $\alpha = 0.4$, $n = 2$.

3.4 Welfare and Comparative Statics

In this subsection, we employ our equilibrium characterization to discuss total social welfare, firm profits/producer surplus as well as consumer surplus and how they depend on the model parameters v_H, v_L, s, α and n. Fortunately, the equilibrium total social welfare is easily obtained: since all prices paid are pure transfers, it is given by the aggregate match values realized through consumption minus the total search costs incurred.

In the "Bertrand region" where Proposition 1 applies, each consumer searches only one random firm, obtains a match value of v_H or v_L with probability α and $1-\alpha$, respectively, and buys there deterministically. Hence, total social welfare in the Bertrand region equals $W_B = \alpha v_H + (1-\alpha)v_L - s$. In all other regions, we have established that each consumer keeps searching until a full match is obtained (if at any firm). In these regions, the aggregate search friction incurred is thus given by[21]

$$S = \left(\sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1} ks\right) + (1-\alpha)^{n-1} ns = s\left[\frac{1-(1-\alpha)^n}{\alpha}\right], \tag{18}$$

[21] Note that for $k = 1, \ldots, n-1$, a fraction $(1-\alpha)^{k-1}\alpha$ of consumers has no full match at the first $k-1$ sampled firms and a full match at the k'th sampled firm, with a per-consumer search cost of ks (first term). A fraction $(1-\alpha)^{n-1}$ of consumers has no full match at the first $n-1$ firms and therefore searches all firms, with a per-consumer search cost of ns (second term).

where the second equality can easily be shown via induction starting from $n = 2$.

At the same time, the realized aggregate match values depend on the equilibrium which is played. In the high-price equilibrium, a fraction $(1-\alpha)^n$ of consumers does not find a full match at any firm and therefore drops out of the market, such that the aggregate match values realized are given by $v_H\left[1-(1-\alpha)^n\right]$. In the low-price equilibrium, once again a fraction $(1-\alpha)^n$ of consumers does not find a full match at any firm, but now these consumers will also buy with their partial match (at the lowest-priced firm). Hence, the aggregate match values realized are given by $v_H\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n$. Finally, in the gap equilibrium, the fraction $(1-\alpha)^n$ of consumers who do not have a full match at any firm will only buy with their partial match if the lowest-priced firm prices below v_L, which happens with probability $1-(1-\kappa)^n$. Hence, the *expected* aggregate match values realized in this case are given by $v_H\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n\left[1-(1-\kappa)^n\right]$. Subtracting the aggregate search friction S from these aggregate match values, the subsequent lemma is immediate.

Lemma 3. *Total social welfare in the market is given by*

$$W=\begin{cases}\alpha v_H+(1-\alpha)v_L-s & \text{if } \alpha\leq\frac{s}{v_H}, \text{ or } \alpha>\frac{s}{v_H} \text{ and } \frac{v_L}{v_H}\geq\bar{\gamma}\\ \left(v_H-\frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right] & \text{if } \alpha>\frac{s}{v_H} \text{ and } \frac{v_L}{v_H}\leq\underline{\gamma}\\ \left(v_H-\frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n\left[1-(1-\kappa)^n\right] & \text{if } \alpha>\frac{s}{v_H} \text{ and } \frac{v_L}{v_H}\in(\underline{\gamma},\tilde{\gamma})\\ \left(v_H-\frac{s}{\alpha}\right)\left[1-(1-\alpha)^n\right]+v_L(1-\alpha)^n & \text{if } \alpha>\frac{s}{v_H} \text{ and } \frac{v_L}{v_H}\in[\tilde{\gamma},\bar{\gamma}).\end{cases} \tag{19}$$

Clearly, welfare losses occur in the high-price and gap equilibrium regions: if all firms were forced to, for example, set some common price weakly below v_L, the mass $(1-\alpha)^n$ of consumers without a full match at any firm would purchase (deterministically instead of probabilistically in the gap-equilibrium region), creating an additional surplus of v_L for each additional consumer served. Moreover, the aggregate search friction would not be affected, since all consumers would still find it optimal to search until they find a full match. We may hence state the following.

Proposition 5. *In the high-price equilibrium ($\alpha>\frac{s}{v_H}$ and $\frac{v_L}{v_H}\leq\underline{\gamma}$), a deterministic welfare loss of $v_L(1-\alpha)^n$ occurs, relative to a situation where firms cannot price above v_L. In the* *gap equilibrium ($\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in (\underline{\gamma}, \tilde{\gamma})$), an expected welfare loss of $v_L(1-\alpha)^n(1-\kappa)^n$ occurs, relative to a situation where firms cannot price above v_L.*[22]

We now turn to comparative statics. Apart from the gap equilibrium, it can easily be seen that the comparative statics of W with respect to the model parameters are monotonic in each equilibrium region: welfare is strictly increasing in v_H and α, weakly increasing in v_L and n (with strict inequality for v_L when not in the high-price region, and strict inequality for n when not in the Bertrand region), and strictly decreasing in s.[23] All of these results are intuitive, as only direct effects are at play: a higher v_H, v_L and α lead to higher (expected) match values realized, a higher n introduces more product variety to generate additional full matches, while a higher s leads to a higher total search friction incurred.

Surprisingly, it turns out that these intuitive comparative statics do not necessarily extend to the gap-equilibrium region. In particular, we can prove analytically[24] that for $n = 2$, there exists an, albeit small, parameter region, close to the boundary to the high-price equilibrium region, where the comparative statics of welfare with respect to v_H and s flip: there, somewhat paradoxically, a marginally higher v_H decreases social welfare, while a marginally higher s increases it. The reason is, that in this specific region, the direct positive effect of a higher v_H or a lower s on welfare is more than outweighed by an indirect negative strategic effect, as firms shift probability mass to prices above v_L in response (κ decreases). Moreover, for $n = 2$, we can show numerically that the same is (sometimes) true for increases in α: close to the boundary to the high-price equilibrium region, marginal increases in α may, but need not, decrease social welfare.[25]

For (discrete) changes in n, what actually matters when assessing the induced change of welfare is how the probability that *at least one* firm prices below v_L, $1-(1-\kappa)^n$, is affected. We have checked, again numerically, that this probability may indeed decrease when n increases. However, even though such a negative strategic effect on welfare through

[22]Note that since prices need to be non-negative, if firms cannot price above v_L, the maximal distance between any two prices is v_L. Even at this maximal distance, consumers would still keep searching a firm with price v_L after having only obtained partial matches if it holds that $v_L \leq v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). One can check that this is automatically satisfied in the high-price equilibrium region, while it only holds if v_L is not too large in the gap-equilibrium region. If v_L is large in the gap-equilibrium region, the second statement in the proposition is only precise if firms' prices do not lie too far apart.

[23]Note that welfare in the low-price region is strictly increasing in α and n since $v_H - \frac{s}{\alpha} > v_L$ in this region.

[24]Details are available from the authors upon request.

[25]For $n \geq 3$, we have not been able to find any (numerical) examples in which the described counterintuitive comparative statics with respect to v_H, s or α prevail.

an increased deadweight loss may occur, our numerical simulations suggest that the positive direct effect through larger product variety always dominates.[26]

Finally, for changes in v_L, the direct and indirect welfare effects point in the same direction, as also firms' equilibrium pricing becomes more aggressive due to decreased production differentiation when v_L increases.

Combining these findings with the fact that welfare is continuous *across* the equilibrium regions, which can easily be verified, enables us to state the following.

Proposition 6. *Total social welfare weakly increases in v_L, and strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$. Moreover, apart from the gap-equilibrium region, social welfare strictly increases in v_H and α, weakly increases in n (and strictly so for $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \bar{\gamma}$), and strictly decreases in s. In the gap-equilibrium region, welfare may be locally decreasing in v_H and α, and locally increasing in s.*

We next consider the comparative statics of firm profits (and producer surplus). For this, effectively only two cases need to be considered, as firms' profits are zero in the Bertrand region, and functionally identical in the low-price and gap equilibrium. Hence, only the low-price/gap equilibrium profits and the high-price equilibrium profits remain, both available in closed form. Inspection of π_H^* (see equation (6)) and π_L^* (see equation (12)) then immediately reveals that firms' expected profits are, whenever they are positive, strictly increasing in v_H, weakly decreasing in v_L (strictly so when not in the high-price region), and strictly decreasing in s. Intuitively, a higher v_H increases product differentiation and the surplus offered to fully-matched consumers, relaxing competition and allowing firms to choose higher prices. A higher v_L intensifies competition by decreasing product differentiation, but also allows the lowest-priced firm to extract more surplus from those consumers who do not have a full match anywhere. However, the former effect always dominates. A higher s depresses prices and profits as consumers become more picky when to search on higher-priced firms, intensifying competition.

[26] Unfortunately, we have not been able to prove this analytically.

Next, as should be expected, increased competition through higher n decreases expected firm profits.[27] Remarkably, the showcased model differs from many standard oligopoly models in the sense that the aggregate firm profits (i.e, producer surplus) may also *increase* in n. Intuitively, this is true because for a relatively low probability of full matches α and a low initial number of firms, the market may expand considerably with entry, as a large number of new consumers (who did not have a full match at any firm previously) may be served (at relatively high prices above v_L).[28]

Finally, it is straightforward to check that firm profits behave non-monotonically in the probability of full matches α. For low values of α, there is Bertrand-type competition, as full matches are so unlikely that consumers do not find it worthwhile to search on after sampling the lowest-priced firm. Starting from a critical threshold, $\alpha \geq \frac{s}{v_H - v_L}$,[29] firms gain market power with increases in α, since there is scope to set prices above marginal cost that will still trigger search by consumers with only partial matches at all lower-priced rival firms (if any). However, as α increases further, product differentiation and firm profits start to decrease again. This is because, as α approaches 1, most consumers will have a full match early on in their search path, leading to strong price competition and low profits as firms attempt to be sampled early by consumers. For $\alpha = 1$, once again Bertrand competition results. Proposition 7 summarizes our findings.

Proposition 7. *Suppose firms make positive profits, $\frac{v_L}{v_H} < \bar{\gamma}$. Then individual and aggregate firm profits strictly increase in v_H, weakly decrease in v_L,*[30] *and strictly decrease in s. They are ambiguous in α. At the same time, individual firm profits strictly decrease in n, while aggregate firm profits are ambiguous in it.*

[27] In the high-price region, this is obvious. In the gap- and low-price regions, this can be seen directly by rewriting

$$\pi_L^* = \frac{\left[(1-\alpha)^n + \alpha\right]\left[\alpha(v_H - v_L) - s\right]}{1 - 2\alpha + \frac{\alpha}{(1-\alpha)^{n-1}}},$$

for which the nominator strictly decreases and the denominator strictly increases in n.

[28] It is easiest to see this in the high-price equilibrium (which, for any given number of firms, is played if $\alpha > \frac{s}{v_H}$ and v_L is sufficiently close to 0). There, the aggregate expected producer surplus with $n+1$ firms, $\Pi_H^*(n+1) = (n+1)\left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^n\alpha$, strictly exceeds the aggregate expected producer surplus with n firms, $\Pi_H^*(n) = n\left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^{n-1}\alpha$, if and only if $(n+1)(1-\alpha)^n > n(1-\alpha)^{n-1}$, which is equivalent to $\alpha < \frac{1}{n+1}$.

[29] This condition is equivalent to $v_L/v_H \leq \bar{\gamma}$.

[30] Strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$.

We conclude this section by briefly considering the comparative statics of consumer surplus. Clearly, despite firms' mixed-strategy pricing, the consumer surplus in the market can easily be obtained indirectly by subtracting the aggregate expected firm profits from total social welfare in each of the different equilibrium regions. Interestingly, it turns out that consumer surplus is ambiguous in v_H, s, α and n.[31] The reason is that, while increases in v_H, α and n or a decrease in s increase consumer surplus for *fixed* prices,[32] this positive effect may be more than offset when firms strategically respond by raising prices. Only for the parameter v_L it can readily be established that consumer surplus always weakly increases in it, and strictly so when not in the high-price equilibrium region.[33] This is because for larger v_L, firms' equilibrium pricing becomes unambiguously more aggressive, while those consumers who do not find a full match anywhere (may) obtain a higher partial match. We formally state these results in Proposition 8.

Proposition 8. *Consumer surplus weakly increases in v_L (and strictly so for $\frac{v_L}{v_H} > \underline{\gamma}$), while it is ambiguous in v_H, s, α and n.*

4 Price-Directed Search and Collusion

Having set out our baseline model, we now turn to the question how firm coordination could affect market outcomes. As argued in the Introduction, collusive behavior should be particularly likely to emerge in the price-transparent markets we consider, as firms can easily monitor their rivals and promptly discipline deviators.

We proceed as follows. First, in Subsection 4.1, we examine tacit collusive agreements. We characterize the payoff-dominant symmetric collusive candidate equilibria across the parameter space and outline their welfare implications. Second, in Subsection 4.2, we con-

[31]For v_H, s and α, this is easily obtained. While numerical simulations suggest that consumer surplus typically rises in n, parameter constellations can be found where it decreases slightly when going from $n = 2$ to $n = 3$. Details are available from the authors upon request.

[32]For an increase in n, this is (for instance) true if one assumes that the newly introduced firm chooses the same price as the highest-priced incumbent. In any case, it is clear that introducing an additional firm while keeping the prices of incumbents fixed cannot decrease consumer surplus.

[33]To see this, recall that total social welfare weakly increases in v_L everywhere (compare with Proposition 6), while aggregate firm profits weakly decrease in v_L everywhere (compare with Proposition 7), and in the former case, this is strict when not in the high-price region.

sider the optimal strategy of an all-inclusive cartel maximizing industry profit, and study its effects on welfare as well.

4.1 Tacit Collusion

We now analyze tacit collusive pricing schemes. We focus on all-inclusive, symmetrical and payoff-dominant collusive pricing sustained by firms using a "grim-trigger" strategy.[34] Whenever we refer to a collusive agreement being "optimal", it is in this specific class of collusive schemes.

Our setup here is an infinitely repeated game in which each stage corresponds to the static game described above. Further, we assume that there is a "new" unit mass of consumers at every stage of the repeated game, and that all "old" consumers leave the market, even if some of them have not been served. Clearly, this excludes anticipation of firms' future pricing on consumers' part, which could in turn influence decisions on current-stage purchases and equilibrium pricing. We also assume that firms evaluate future profits according to a common discount factor $\delta \in (0,1)$.

First, we show that, depending on the market fundamentals, firms would like to coordinate on different optimal collusive prices.

Proposition 9. *The payoff-dominant, symmetric collusive price is given by*[35]

$$p^C = \begin{cases} \alpha v_H + (1-\alpha)v_L - s \le v_L & \textit{if } \alpha \le \frac{s}{v_H}, \textit{ or } \alpha > \frac{s}{v_H} \textit{ and } \frac{v_L}{v_H} \ge \bar{\gamma} \\ v_L & \textit{if } \alpha > \frac{s}{v_H} \textit{ and } \frac{v_L}{v_H} \in [\gamma^C, \bar{\gamma}) \\ v_H - \frac{s}{\alpha} > v_L & \textit{if } \alpha > \frac{s}{v_H} \textit{ and } \frac{v_L}{v_H} \in [0, \gamma^C), \end{cases} \tag{20}$$

where

$$\gamma^C := \left(1 - \frac{s}{\alpha v_H}\right)[1-(1-\alpha)^n] \in (\underline{\gamma}, \bar{\gamma}). \tag{21}$$

[34] We can easily characterize also other strategy combinations to enforce cooperation, such as "stick-and-carrot" strategies (i.e., optimal penal codes, see Abreu (1988)). Such more complex strategies are able to sustain collusion on the optimal price level for a larger set of discount factors δ. However, they clearly make no difference on the equilibrium path whenever collusion can also be sustained with grim-trigger strategies.

[35] In what follows, we use the tie-breaking rule that firms coordinate on the price level which induces higher demand in case they are indifferent between two price levels. This implies that for $\frac{v_L}{v_H} = \gamma^C$, firms coordinate on v_L.

Collusion on this price level can be sustained using grim-trigger strategies if and only if firms' discount factor δ *is sufficiently close to* 1.

Proof. See Appendix A. □

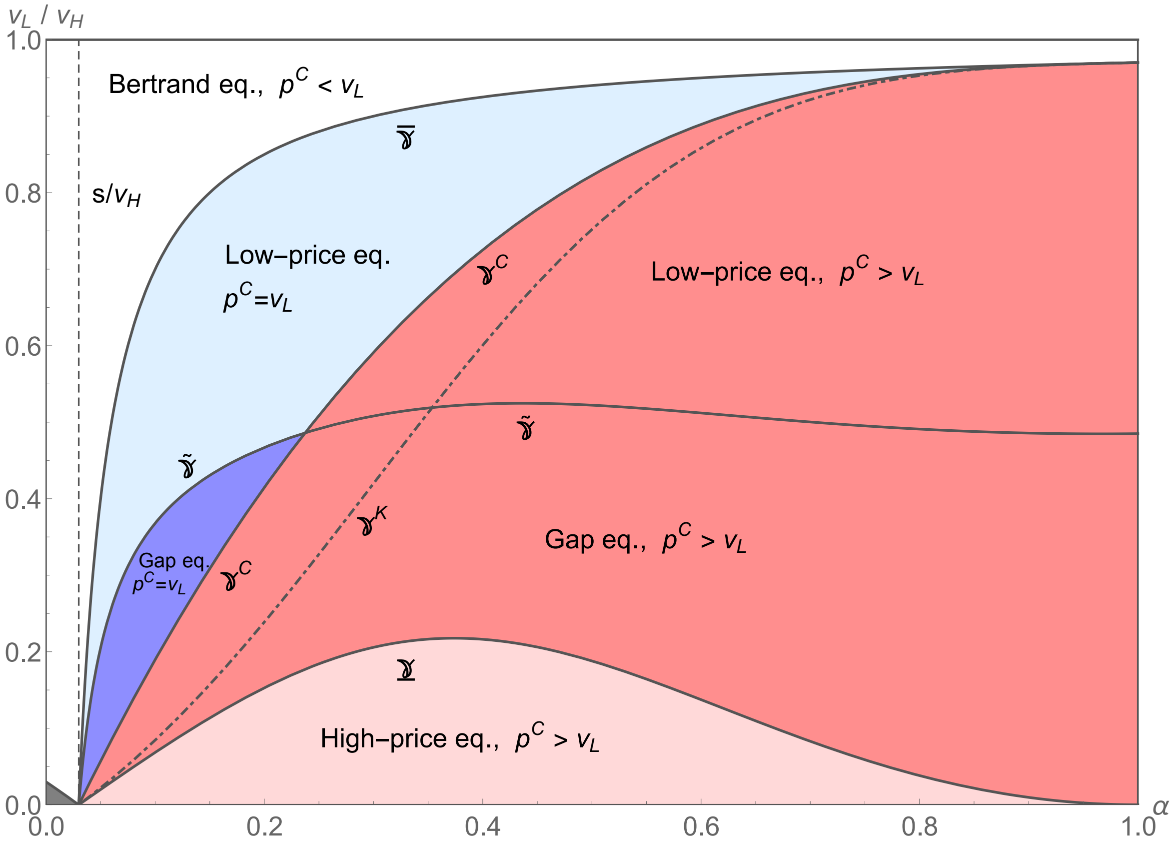

Figure 3: Pricing and welfare comparison between the optimal symmetric collusive schemes and the equilibria of the baseline model. The parameters used are $\frac{s}{v_H} = 0.03$ and $n = 3$.

Figure 3 provides a graphical representation of the different optimal collusive schemes partitioning the parameter-space (it also contains a welfare comparison to the baseline model which will be explained below). For now, in the white region above $\bar{\gamma}$, the optimal symmetric collusive price is $p^C = \alpha v_H + (1-\alpha)v_L - s < v_L$, in the light blue *and* dark blue regions between γ^C and $\bar{\gamma}$, the optimal price is $p^C = v_L$, while in the light red *and* dark red regions below γ^C, the optimal price is $p^C = v_H - \frac{s}{\alpha} > v_L$.

We will now explain the intuition behind these different cases and their welfare implications. Recall first that if full matches are quite unlikely, or when consumers' valuations of full and partial matches are relatively close ($\alpha \leq s/v_H$, or $\alpha > s/v_H$ and $v_L/v_H \geq \bar{\gamma}$), firms face a Bertrand-type competition. Consumers search at most once, and only for prices below v_L, then buy regardless of whether a full or partial match is discovered. In this case, the optimal collusive agreement involves all firms charging a price corresponding to the expected gross utility of a consumer searching exactly one firm $(\alpha v_H + (1-\alpha)v_L - s)$. Still, the whole market is served, such that no deadweight loss results from collusion. The only consequence is a redistribution of rents from consumers to firms (white region in Figure 3).

When not in the Bertrand region, the highest price which firms can coordinate on is the highest price which keeps consumers in the market, $p_{max} = v_H - \frac{s}{\alpha} > v_L$. Firms should optimally set this price when consumers' valuation for partial matches is relatively low, $v_L/v_H < \gamma^C$. The consequences for welfare are clear. Whenever, in the corresponding equilibrium of the baseline model, there is a positive probability of prices being so low that consumers with only partial matches at every firm are served, $v_L/v_H \in (\underline{\gamma}, \gamma^C)$, the collusive agreement leads to a deadweight welfare loss (and a redistribution of consumer rents to firms). This is because the collusive price lies above v_L deterministically (dark-red region in Figure 3). Otherwise, for $v_L/v_H \leq \underline{\gamma}$, the consequence of collusion is, again, a mere redistribution of consumer rents (light-red region in Figure 3).

However, when with $v_L/v_H \in [\gamma^C, \bar{\gamma})$ product heterogeneity is intermediate (partial matches are neither very low compared to full matches, but also not almost as high) – or alternatively, for fixed v_L/v_H, the probability of full matches is intermediate[36] – it is no longer most profitable for firms to coordinate on a price higher than v_L. This is because by excluding consumers who do not find a full match at any firm, substantial revenue losses would be incurred. Firms instead optimally coordinate on the highest price level that is low enough to guarantee that the whole market is served ($p^C = v_L$). This obviously has no effect on total welfare when all firms also set such low prices with probability one in the corresponding one-shot Nash equilibrium, which holds if $v_L/v_H \geq \tilde{\gamma}$. In this case, consumer rents are again clearly transfered to firms (light-blue region in Figure 3).

But, as one of our main findings, it can be seen that a collusive coordination on v_L can also be optimal when firms set prices below v_L only with positive probability in the equi-

[36]Note that $\gamma^C(\alpha)$ is strictly increasing in α over the relevant range, with $\gamma^C(\frac{s}{v_H}) = 0$ and $\gamma^C(1) = 1 - \frac{s}{v_H}$. Moreover, $\gamma^C < \bar{\gamma}$. Hence, for any $v_L/v_H \in (0, 1 - \frac{s}{v_H})$, colluding on $p^C = v_L$ will be optimal if and only if α is intermediate (compare with Figure 3). An alternative condition for optimal collusion on v_L is therefore that the probability of full matches needs to be relatively low, but not so low to end up in the Bertrand region.

librium of the one-shot game (the gap equilibrium), which is the case for $v_L/v_H \in [\gamma^C, \tilde{\gamma})$, thereby *reducing* deadweight loss. The overall implication is that welfare in such markets may actually *increase* when firms collude (dark-blue region in Figure 3).[37] We moreover show that for any number of firms, there is a parameter region where this is the case. Proposition 10 formally summarizes our findings.

Proposition 10. *Suppose that $\alpha > \frac{s}{v_H}$.[38] Compared to the baseline model, when the optimal collusive price level can be supported in equilibrium, total social welfare remains constant and consumer rents are redistributed to firms for $\frac{v_L}{v_H} \leq \underline{\gamma}$ or $\frac{v_L}{v_H} \geq \max\{\tilde{\gamma}, \gamma^C\}$. Welfare and consumer surplus strictly decrease for $\frac{v_L}{v_H} \in (\underline{\gamma}, \gamma^C)$. Welfare strictly increases for $\frac{v_L}{v_H} \in [\gamma^C, \tilde{\gamma})$, while consumer surplus may decrease or increase. For any number of firms, there exists a parameter region where $\tilde{\gamma} > \gamma^C$, such that welfare may indeed increase through collusion.*

Proof. See Appendix A. □

To gain some intuition why the payoff-dominant symmetric collusive equilibrium may increase welfare, relative to the one-shot Nash equilibrium, consider a situation where with $\alpha > s/v_H$ product differentiation can be sufficiently large (for v_L sufficiently small) such that there is not always Bertrand pricing in the baseline equilibrium. Recall then that there are two conditions for collusion to increase welfare. First, product differentiation needs to be sufficiently high, $v_L/v_H < \tilde{\gamma}$, such that with unrestrained competition, firms would not always price below v_L – otherwise, welfare clearly cannot improve through collusion. For α sufficiently close above s/v_H, it can now be shown that a higher α, by giving firms scope to price above marginal cost, relaxes this condition. But second, v_L also needs to be sufficiently large, $v_L/v_H \geq \gamma^C$, such that firms find it optimal to collude on v_L rather than on the highest possible price $v_H - \frac{s}{\alpha} > v_L$. This second condition becomes harder to satisfy as α increases, as both $v_H - \frac{s}{\alpha}$ and the corresponding demand $1-(1-\alpha)^n$ strictly increase in α. Still, for values of α that are not too far above s/v_H, there may exist a range of v_L/v_H such that both conditions are satisfied. In particular, we can show that this is always true for search costs that are sufficiently close to zero. If this is the case, collusion indeed increases welfare.

[37] We can even find parameter constellations in this region where also consumer rents increase in face of firms' coordination. However, this only seems to be possible for duopoly, and only if α and v_L/v_H are very small. Details can be obtained from the authors upon request.

[38] For $\alpha \leq \frac{s}{v_H}$, total welfare remains constant, and only consumer rents are transferred to firms.

As an additional result, note that if product differentiation is not too large (with $v_L/v_H \geq \gamma^C$) and firms already optimally collude on v_L, a small decrease in search costs may actually lead to a significant *decrease* in total welfare. This is because a decrease in s makes collusion on the highest possible price level $v_H - \frac{s}{\alpha}$ relatively more attractive than colluding on v_L, such that the optimal collusive price may switch to the former. Welfare then decreases if the drop in aggregate search costs through lower s is more than offset by the additional deadweight loss caused by non-fully-matched consumers dropping out of the market. In particular, a marginal decrease of s starting from $v_L/v_H = \gamma^C$ decreases welfare by $v_L(1-\alpha)^n$, such that by continuity, also a sufficiently small *discrete* decrease from s to $s' < s$, starting from v_L/v_H close above $\gamma^C(s)$ and leading to $v_L/v_H < \gamma^C(s')$, will decrease welfare.

4.2 Cartelization

We finally consider firms joining an all-inclusive cartel (or alternatively, merging into one multi-product retailer). We assume that the total cartel profits are divided equally among its members. As we will show, the profit-maximizing strategy of such a cartel can be different from some symmetric collusive agreements, generating a higher industry profit and leading to different welfare implications.

But clearly, in the Bertrand region, where either $\alpha \leq s/v_H$, or $\alpha > s/v_H$ and $v_L/v_H \geq \bar{\gamma}$, a cartel's profit-maximizing strategy coincides with the optimal symmetric collusive agreement, since consumers only ever search once, and the highest price that induces search is given by $\alpha v_H + (1-\alpha)v_L - s \leq v_L$.[39] Similarly, when not in the Bertrand region, and if products are sufficiently differentiated with v_L/v_H sufficiently small, the cartel will, like in the optimal symmetric collusive agreement, find it most profitable to set the highest possible price that still induces search for every product, thus losing out on consumers who do not discover a full match at any firm. If so, the profit-maximizing strategy of the cartel is unique, and prices are identical to those in Proposition 9 (each cartel member prices at $v_H - \frac{s}{\alpha} > v_L$). We show in the proof of the subsequent Proposition 11 that this is the case if and only if $v_L/v_H < \gamma^K := \left(1 - \frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha+(1-\alpha)^n}$.

[39] However, the associated cartel profit could also be achieved differently, e.g. by having only *one* firm set the highest possible price which induces search by all consumers, while the other firms could set arbitrarily higher prices.

However, if the ratio of consumers' valuations takes on intermediate values, $v_L/v_H \in [\gamma^K, \bar{\gamma})$ – or, alternatively, if the probability of full matches α is intermediate – it is optimal to charge the low price v_L for *exactly one* product and the highest possible price $v_H - \frac{s}{\alpha}$ that keeps consumers in the market for all other products.[40] By providing the former "compromise option", firms extract as much consumer rent as possible while *still* serving all consumers who do not find a full match at any firm. This is particularly attractive if full matches are relatively rare. Obviously, setting the price of more than one product to v_L cannot be optimal, since total demand would be unchanged, but a larger fraction of consumers would buy at this low price.[41]

Note that if the cartel finds it most profitable to serve the whole market by pricing one product at v_L, the same implications for welfare hold as in Proposition 10 when $p^C \leq v_L$. Hence, welfare remains constant whenever the whole market would be served in the equilibrium of the baseline game as well, while it strictly increases otherwise. However, it is easily shown that the parameter region in which total welfare increases is strictly larger (in the sense of set inclusion) than under the optimal symmetric collusive scheme.[42] Intuitively, whenever it is the most profitable *symmetric* collusive strategy to serve the whole market at a price of v_L, rather than to serve only part of the market at $v_H - \frac{s}{\alpha} > v_L$, setting the price of only one product to v_L (such that all consumers who do not find a full match at any firm stay in the market) while charging the highest possible price for all other products must be even more profitable in comparison – the strategy increases the average mark-up without losing out on demand. Hence, the range of parameters where firms find it optimal to price one product at v_L (rather than to collectively price at $v_H - \frac{s}{\alpha} > v_L$) is larger than the range of parameters where firms find it optimal to symmetrically collude at v_L (again, rather than to collectively price at $v_H - \frac{s}{\alpha} > v_L$). In Figure 3, one can observe that the region where

[40] Similar to the reasoning in Footnote 36 above, note that $\gamma^K(\alpha)$ is strictly increasing in α over the relevant range, with $\gamma^K(\frac{s}{v_H}) = 0$ and $\gamma^K(1) = 1 - \frac{s}{v_H}$. Moreover, $\gamma^K < \bar{\gamma}$. Hence, for any $v_L/v_H \in (0, 1 - \frac{s}{v_H})$, it is optimal for the cartel to price one product at v_L and all others at $v_H - \frac{s}{\alpha}$ if and only if α is intermediate (compare once more with Figure 3). Hence, the probability of full matches needs to be relatively low, but not so low to end up in the Bertrand region.

[41] Let $1 \leq m \leq n$ products' prices be at v_L, and the remaining $n - m$ prices at $v_H - \frac{s}{\alpha}$. Then, since only those consumers who have no full match at any of the m low-priced firms *and* have a full match at some of the $n - m$ high-priced firms buy from the latter, while all others buy from the former, the cartel makes a profit of $\pi = \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)^m[1-(1-\alpha)^{n-m}] + v_L[1-(1-\alpha)^m[1-(1-\alpha)^{n-m}]] = v_L + \left(v_H - \frac{s}{\alpha} - v_L\right)[(1-\alpha)^m - (1-\alpha)^n]$. Since in the "non-Bertrand" region $v_H - \frac{s}{\alpha} > v_L$, profits are clearly maximal for $m = 1$.

[42] The statement boils down to $\gamma^C > \gamma^K \Leftrightarrow 1-(1-\alpha)^n > \frac{\alpha}{\alpha+(1-\alpha)^n} \Leftrightarrow 1-\alpha-(1-\alpha)^n > 0$, which is true.

welfare increases through firm coordination would expand from the dark-blue region to the region between γ^K and $\tilde{\gamma}$. We now state our findings formally.

Proposition 11. *Suppose that* $\alpha > \frac{s}{v_H}$ *and* $\frac{v_L}{v_H} < \bar{\gamma}$.[43] *Then, if* $\frac{v_L}{v_H} \in [0, \gamma^K)$, *where* $\gamma^K := \left(1 - \frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha+(1-\alpha)^n} \in (\underline{\gamma}, \gamma^C)$, *the all-inclusive cartel's profit-maximizing strategy has all firms setting their price at* $v_H - \frac{s}{\alpha}$. *If instead* $\frac{v_L}{v_H} \in [\gamma^K, \bar{\gamma})$, *the all-inclusive cartel's profit-maximizing strategy involves firms setting prices such that*

$$p_i^K := v_L \qquad \text{for one } i \in \{1, \ldots, n\}$$
$$p_j^K := v_H - \frac{s}{\alpha} \qquad \forall j \neq i.$$

For $\frac{v_L}{v_H} \in [\gamma^K, \gamma^C)$, *total social welfare and consumer surplus strictly increase, compared to the payoff-dominant collusive equilibrium. For* $\frac{v_L}{v_H} \in [\gamma^K, \tilde{\gamma})$, *total social welfare strictly increases under cartelization, compared to the non-cooperative outcome.*

Proof. See Appendix A. □

In principle, we could also allow firms to choose the above asymmetric strategy combination without forming a cartel, but tacitly coordinating on this behavior might arguably be substantially more difficult to sustain.

5 Conclusion

We have set up a tractable model of price-directed search in which consumers observe prices, but need to engage in costly sequential search in order to find out whether products fully or only partially match their needs. We have characterized the set of symmetric equilibria and show that welfare losses may occur, as all firms may (deterministically or stochastically) price above consumers' valuation for partial matches. If this happens, part of the consumers inefficiently drop out of the market. Analyzing collusion and cartelization, we find that, perhaps surprisingly, social welfare may in fact increase in face of such coordination. This is particularly likely when search costs are low, products are sufficiently differentiated, and

[43] If $\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \bar{\gamma}$, the cartel solution has all firms pricing at $p^K = \alpha v_H + (1-\alpha)v_L - s \leq v_L$, welfare is unaffected relative to the baseline game, while all consumer surplus is transferred to firms.

consumers have moderately "picky" tastes such that the likelihood of full matches is not too high. One potential implication for policymakers is that firm coordination on relatively low prices, in particular in (online) search markets where consumers derive some baseline utility from products but the incidence of very good matches is relatively low, may be treated more benevolently, as unrestricted competition may even lead to worse market outcomes.

References

Dilip Abreu. On the Theory of Infinitely Repeated Games with Discounting. *Econometrica*, 56(2): 383–396, March 1988.

Simon P. Anderson and Régis Renault. Pricing, Product Diversity, and Search Costs: A Bertrand-Chamberlin-Diamond Model. *RAND Journal of Economics*, 30(4):719–735, Winter 1999.

Simon P. Anderson and Régis Renault. Firm pricing with consumer search. In *Handbook of Game Theory and Industrial Organization, Volume II*, Chapters, chapter 8, pages 177–224. Edward Elgar Publishing, February 2018.

Maria Arbatskaya. Ordered search. *RAND Journal of Economics*, 38(1):119–126, March 2007.

Mark Armstrong. Ordered Consumer Search. *Journal of European Economic Association*, 15(5): 989–1024, 2017.

Mark Armstrong and Jidong Zhou. Paying for prominence. *The Economic Journal*, 121:368–395, 2011.

Mark Armstrong, John Vickers, and Jidong Zhou. Prominence and consumer search. *RAND Journal of Economics*, 40(2):209–233, 2009.

Susan Athey and Glenn Ellison. Position auctions with consumer search. *Quarterly Journal of Economics*, 126:1213–1270, 2011.

Michael Baye, J. Rupert J. Gatti, Paul Kattuman, and John Morgan. Clicks, discontinuities, and firm demand online. *Journal of Economics & Management Strategy*, 18(4):935–975, 2009.

Michael R. Baye, John Morgan, and Patrick Scholten. Information, Search, and Price Dispersion. In *Handbook in Economics and Information Systems (ed. T. Hendershott)*, volume 1, chapter 6. Amsterdam: Elsevier, 2006.

Colin Campbell, Gautam Ray, and Waleed A. Muhanna. Search and Collusion in Electronic Markets. *Management Science*, 51(3):1497–1507, 2005.

Yongmin Chen and Chuan He. Paid placement: advertising and search on the internet. *The Economic Journal*, 121:F309–F328, 2011.

Michael Choi, Anovia Y. Dai, and Kyungmin Kim. Consumer Search and Price Competition. *Econometrica*, 86(4):1257–1281, 2018.

Yucheng Ding and Tianle Zhang. Price-directed consumer search. *International Journal of Industrial Organization*, 58:106–135, 2018.

Glenn Ellison and Sara Fisher Ellison. Search, obfuscation, and price elasticities on the internet. *Econometrica*, 77(2):427–452, 2009.

Switgard Feuerstein. Collusion in Industrial Economics–A Survey. *Journal of Industry, Competition and Trade*, 5(3):163–198, December 2005.

Marco Haan, José L. Moraga-González, and Vaiva Petrikaite. A Model of Directed Consumer Search. *International Journal of Industrial Organization*, 61:223–255, 2018.

Norman J. Ireland. The Provision of Information in a Bertrand Oligopoly. *Journal of Industrial Economics*, 41(1):61–76, March 1993.

José L. Moraga-González. Consumer Search in Digital Markets. Survey, Vrije Universiteit Amsterdam and University of Groningen, 2018.

José L. Moraga-González, Zsolt Sandor, and Matthijs Wildenbeest. Consumer Search and Prices in the Automobile Market. CEPR Discussion Papers 10487, C.E.P.R. Discussion Papers, March 2015. URL `https://ideas.repec.org/p/cpr/ceprdp/10487.html`.

Arvid Nilsson. Transparency and Competition. SSE/EFI Working Paper Series in Economics and Finance 298, Stockholm School of Economics, February 1999. URL `https://ideas.repec.org/p/hhs/hastef/0298.html`.

Mitsukuni Nishida and Marc Remer. Lowering consumer search costs can lead to higher prices. *Economics Letters*, 162(C):1–4, 2018.

Vaiva Petrikaitė. Collusion with costly consumer search. *International Journal of Industrial Organization*, 44(C):1–10, 2016.

Christian Schultz. Transparency on the consumer side and tacit collusion. *European Economic Review*, 49(2):279–297, February 2005.

Christian Schultz. Collusion in Markets with Imperfect Price Information on Both Sides. *Review of Industrial Organization*, 50(3):287–301, May 2017.

Jian Shen. Ex-Ante Preference in a Consumer Search Market. Technical report, SSRN, April 2015. URL `https://ssrn.com/abstract=2826011`.

Dale O. Stahl. Oligopolistic Pricing with Sequential Consumer Search. *American Economic Review*, 79(4):700–712, September 1989.

Hal R. Varian. A Model of Sales. *American Economic Review*, 70(4):651–59, September 1980.

Asher Wolinsky. True Monopolistic Competition as a Result of Imperfect Information. *The Quarterly Journal of Economics*, 101(3):493–511, August 1986.

Jidong Zhou. Ordered search in differentiated markets. *International Journal of Industrial Organization*, 29(2):253–262, March 2011.

6 Appendix A: Technical Proofs

Proof of Proposition 2. We first give a detailed existence proof. We then provide a sketch how uniqueness can be established.

Existence. Given that all other firms sample prices from the CDF $F_H(p)$ as defined in equation (3), it is first easy to see that for any price in the candidate equilibrium's support $[\underline{p}_H, \overline{p}_H]$, it indeed holds that $\pi_i(p) = p(1-\alpha F_H(p))^{n-1}\alpha = \pi_H^*$, with π_H^* as defined in equation (6). It is moreover straightforward to check that given the imposed parameter restrictions $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \in [0, \underline{\gamma}]$, $F_H(p)$ is strictly increasing in its support, and that $\underline{p}_H > v_L$. Hence, the candidate equilibrium is well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. We start by proving that there are no profitable deviations below the lowest price $\underline{p}_H$. For this, note first that it clearly cannot be optimal to deviate to any price in the range $(v_L, \underline{p}_H)$, as the same demand would already be achieved when pricing at $\underline{p}_H$. Note next that when deviating to v_L, a firm would make an expected profit of $\pi_i(v_L) = v_L[\alpha + (1-\alpha)^n]$, as it would become the lowest-priced firm that is sampled first with certainty, attracting all of its fully-matched consumers as well as all consumers with no full match at any firm (who would eventually return to the deviating firm after having searched all firms). Note moreover that those consumers who are only partially matched at the deviating firm would always continue to search, since even if all rival firms priced at $\overline{p}_H$, the expected gains from search would be non-negative. Given that $\alpha > \frac{s}{v_H}$ as assumed, it is then easy to see that $\pi_i(v_L) \leq \pi_H^*$ if and only if $\frac{v_L}{v_H} \leq \underline{\gamma}$, as also assumed.

We next establish that under the relevant parameter restrictions, it is never profitable to price below v_L, as the expected profits for any deviation price $p \in (0, v_L)$ are lower than when deviating to v_L. To see this, note that since the deviating firm is guaranteed to be searched first, the fraction α of consumers who find a full match at this firm will immediately buy there. Furthermore, consumers who only find a partial match will only search those rival firms j (and buy there in case they find a full match) whose price difference is not too large relative to the deviation price, that is, for which $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). The probability that one rival sets $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (such that it will be searched) and provides a full match (such that it will attract the deviating firm's partially-matched consumers) is given by $F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha$. Hence, the probability that *not a single* rival firm does so is given by $\left[1 - F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}$. In turn, the expected deviation profits for $p \leq v_L$ can be written as

$$\begin{aligned}\pi_i(p) &= p\left[\alpha + (1-\alpha)\left[1 - F_H\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right] \\ &= p\left[\alpha + (1-\alpha)^n\left(\frac{v_H - \frac{s}{\alpha}}{p + v_H - v_L - \frac{s}{\alpha}}\right)\right].\end{aligned}$$

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p when $v_H - v_L - \frac{s}{\alpha} > 0$ (as holds in the considered parameter region), it is easy to see that the last expression is strictly increasing in p. It is thus indeed maximized for $p = v_L$, such that deviations below v_L cannot be optimal.

It remains to show that there is no profitable deviation above $\overline{p}_H = v_H - \frac{s}{\alpha}$. But since no firm would ever be searched for $p > \overline{p}_H$ (compare once again with Lemma 1), this is immediately evident. This completes the proof of existence.

Uniqueness. For brevity, we only provide a sketch how uniqueness can be established in the class of symmetric equilibria. This sketch also applies for the subsequent Propositions 3 and 4.

Note first that the parameter requirement for Proposition 2 (as well as Propositions 3 and 4) is that $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \overline{\gamma}$, which is equivalent to $\alpha(v_H - v_L) > s$. Only in this case, consumers may have an incentive to search on after discovering only a partial match at the lowest-priced firm, avoiding the Bertrand outcome as unique symmetric equilibrium.

Second, since the parameter requirement $\alpha(v_H - v_L) > s$ is equivalent to $v_H - \frac{s}{\alpha} > v_L$, it follows immediately from consumers' optimal search rule in Lemma 1 that no firm can make a positive profit when pricing strictly above $p_{max} := v_H - \frac{s}{\alpha}$, as it would never be searched. But clearly, each firm can guarantee a positive profit by pricing at $v_H - v_L - \frac{s}{\alpha} > 0$, since it would be searched by only partially-matched consumers even for $p_1 = 0$. Hence, no firm may ever price above p_{max} in equilibrium.

Third, given that $\alpha(v_H - v_L) > s$, clearly no symmetric pure-strategy equilibrium can exist, as marginally undercutting any symmetric candidate equilibrium price $p^* \in (0, v_H - \frac{s}{\alpha}]$ would give a firm a discretely higher profit (by being searched first by all consumers). By a similar logic, there can be no mass points in any symmetric equilibrium.

Denoting $\overline{p}$ and $\underline{p}$ as the upper and lower support bound of any symmetric candidate equilibrium, with $\overline{p} \leq p_{max}$, the crucial steps are now to establish that either (i) $\overline{p} - \underline{p} < \Delta := v_H - v_L - \frac{s}{\alpha}$ and $\overline{p} = p_{max}$ or (ii) $\overline{p} - \underline{p} = \Delta$. The former is trivial to see by contradiction: in any candidate equilibrium where $\overline{p} - \underline{p} < \Delta$ and $\overline{p} < p_{max}$, a firm choosing $p_i = \overline{p}$ could unilaterally increase its profit by choosing $p_i = \min\{\underline{p} + \Delta, p_{max}\}$ instead. This gives the firm an identical demand of $(1-\alpha)^{n-1}\alpha$ at a higher price (in particular, since there can be no mass point at $\overline{p}$).

It is significantly more demanding to show that $\overline{p} - \underline{p} > \Delta$ cannot hold. This can be proven by contradiction via the following steps: (1) For $\overline{p} - \underline{p} > \Delta$, it must hold that the density $f(\overline{p}) \stackrel{!}{=} 0$ by comparing $\lim_{p \uparrow \overline{p}} \pi_i'(p)$ with $\lim_{p \downarrow \overline{p}} \pi_i'(p)$, (2) from this, it follows that the density $f(\overline{p} - \Delta) > 0$, such that $\overline{p} - \Delta$ must lie in the equilibrium support, (3) $\lim_{p \uparrow (\overline{p} - \Delta)} \pi_i'(p) = \lim_{p \downarrow (\overline{p} - \Delta)} \pi_i'(p)$ as a consequence of $f(\overline{p}) = 0$, (4) combining the conditions $\pi_i'(\overline{p}) \stackrel{!}{=} 0$ and $\pi_i'(\overline{p} - \Delta) \stackrel{!}{=} 0$[44] and finally observing that this leads to a contradiction.

Using the result that either (i) $\overline{p} - \underline{p} < \Delta$ and $\overline{p} = p_{max}$ or (ii) $\overline{p} - \underline{p} = \Delta$, the required profit indifference at $\overline{p}$ and $\underline{p}$ gives rise to a respectively unique solution for $\overline{p}$, $\underline{p}$ and the candidate equilibrium profit π^*, both for (i) (as provided in equations (4), 5) and (6)) and (ii) (as provided in equations (12), (13) and (14)). The corresponding $\overline{p}$ for (ii) is however not compatible with $\overline{p} \leq p_{max}$ if $\frac{v_L}{v_H} \leq \underline{\gamma}$,[45] as

[44] The latter must be true since $\overline{p} - \Delta$ lies in the equilibrium support, such that there must also be probability mass immediately below or above $\overline{p} - \Delta$ (or both).

[45] In the borderline case where $\frac{v_L}{v_H} = \underline{\gamma}$, it actually holds that $\overline{p} = p_{max}$ and $\underline{p} = v_L$. In particular, this would mean that the firms choose prices weakly lower than v_L with zero probability, yet $p = v_L$ lies in the support, with discretely higher demand than when setting $p + \varepsilon$ for any $\varepsilon > 0$ (due to returning demand). Hence, there would need to be a gap in the equilibrium distribution for prices slightly above v_L, which is however incompatible with $F(v_L) = 0$ and v_L being part of the support.

assumed for Proposition 2 (while it is compatible with it for $\frac{v_L}{v_H} \in (\underline{\gamma}, \overline{\gamma})$, in which case the candidate equilibrium following (i) does not exist). Noting finally that with $\overline{p} - \underline{p} \leq \Delta$ there can be no holes in the equilibrium support apart possibly from some range immediately above v_L, in each case the respective equilibrium follows uniquely from construction. $\square$

Proof of Proposition 3. In what follows, we prove existence. For uniqueness, the argument at the end of the proof of Proposition 2 applies.

Existence. It is first easy to verify that simultaneously solving equations (7) and (8) gives $\underline{p}_L$, $\overline{p}_L$ and $\pi_L^* = \overline{p}_L(1-\alpha)^{n-1}\alpha$ as reported in the proposition. Moreover, since $\alpha(v_H - v_L) - s > 0$ in the considered parameter region, these objects are all strictly positive, with clearly $\overline{p}_L > \underline{p}_L$, and $\overline{p}_L \leq v_L$ since by assumption $v_L/v_H \geq \tilde{\gamma}$. By construction, the implicit definition of $F_L(p)$ in equation (11) ensures that all prices in the candidate equilibrium's support yield the same expected profit. One may also note from equation (11) that $F_L(p)$ is strictly increasing in its support. Hence, all equilibrium objects are well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. First, we show that there is no profitable deviation above $\overline{p}_L$. A deviating firm pricing at some $p > \overline{p}_L$ will only be searched if its price is not too high relative to the lowest-priced firm, which holds if $p_1 \geq p - \left(v_H - v_L - \frac{s}{\alpha}\right)$ (compare with Lemma 1). Equivalently, in order for the deviating firm to be searched at all, all rival firms' prices must lie above $p - \left(v_H - v_L - \frac{s}{\alpha}\right)$. Then, the deviating firm will cater to the mass $(1-\alpha)^{n-1}\alpha$ consumers who don't have a full match at any rival firm, but a full match at this firm. Thus, the expected profit at any such price $p > \overline{p}_L$ can be written as

$$\pi_i(p) = p\left[1 - F_L\left(p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha. \tag{22}$$

For prices which lie in the support of the candidate equilibrium, i.e. $p \in [\underline{p}_L, \overline{p}_L]$, the expected profit is by construction equal to π_L^*, where we replicate here the implicit definition of $F_L(p)$, equation (11), for convenience:

$$\pi_i(p) = p\left[\left(1 - \alpha F_L(p)\right)^{n-1}\alpha + \left(1 - F_L(p)\right)^{n-1}(1-\alpha)^n\right] = \pi_L^*. \tag{23}$$

Since $F_L(p)$ cannot be obtained in closed form for an arbitrary number of firms n, we will use an estimation. Rewriting (23), it holds for $p \in [\underline{p}_L, \overline{p}_L]$ that

$$(1-F_L(p))^{n-1} = \frac{\frac{\pi_L^*}{p} - (1-\alpha F_L(p))^{n-1}\alpha}{(1-\alpha)^n} \leq \frac{\frac{\pi_L^*}{p} - (1-F_L(p))^{n-1}\alpha}{(1-\alpha)^n},$$

such that by isolating $(1-F_L(p))^{n-1}$ we obtain

$$(1-F_L(p))^{n-1} \leq \frac{\pi_L^*}{p[\alpha+(1-\alpha)^n]}. \tag{24}$$

For $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$, it holds that $p-(v_H - v_L - \frac{s}{\alpha}) \in [\underline{p}_L, \overline{p}_L]$. Hence, by inequality (24), we have that for $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$,

$$\left[1-F_L\Big(p-\big(v_H - v_L - \frac{s}{\alpha}\big)\Big)\right]^{n-1} \leq \frac{\pi_L^*}{\left[p-\big(v_H - v_L - \frac{s}{\alpha}\big)\right][\alpha+(1-\alpha)^n]}.$$

In turn, this implies that the following estimation can be given for equation (22) and $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$:

$$\begin{aligned}
\pi_i(p) &= p\left[1-F_L\left(p-\left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha \\
&\leq p\left[\frac{\pi_L^*}{\left[p-\big(v_H - v_L - \frac{s}{\alpha}\big)\right][\alpha+(1-\alpha)^n]}\right](1-\alpha)^{n-1}\alpha.
\end{aligned}$$

Since $\frac{p}{p-(v_H-v_L-\frac{s}{\alpha})}$ is strictly decreasing in p for $v_H - v_L - \frac{s}{\alpha} > 0$ as assumed for the proposition, the last expression is thereby maximized for $p = \overline{p}_L$. This implies that for $p \in [\overline{p}_L, \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})]$,[46]

$$\pi_i(p) \leq \overline{p}_L\left[\frac{\pi_L^*}{\left[\overline{p}_L-\big(v_H - v_L - \frac{s}{\alpha}\big)\right][\alpha+(1-\alpha)^n]}\right](1-\alpha)^{n-1}\alpha = \pi_L^*.$$

Hence, deviations above $\overline{p}_L$ are indeed not profitable.

Next, we show that there is no profitable deviation below $\underline{p}_L$. For such low prices, there is now a positive probability that some or all rival firms draw high enough prices such that consumers who are only partially matched at the deviating firm do not search them. Precisely, for deviation prices $p < \underline{p}_L$, consumers that are only partially matched at the deviating firm will only search rival firms j for which $p_j \leq p + v_H - v_L - \frac{s}{\alpha}$ (compare with Lemma 1). Moreover, consumers will only buy at such firms if they are fully matched at them. The probability to lose the mass $1-\alpha$ of partially-matched consumers

[46]For $p > \overline{p}_L + (v_H - v_L - \frac{s}{\alpha})$, $\pi_i(p) = 0$, since no consumer would ever search the deviating firm.

towards a single rival is therefore given by $F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha$. Consequently, the probability *not* to lose these consumers against *any* rival firm is given by $\Big[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\Big]^{n-1}$. Hence, we can write a deviating firm's expected profit for $p<\underline{p}_L$ as

$$\pi_i(p)=p\left[\alpha+(1-\alpha)\left[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\right]^{n-1}\right]. \tag{25}$$

Again, our strategy will be to use an estimation for the additional expected demand, which will be derived from the only implicitly defined CDF F_L. Using once more equation (23), we find that for $p\in[\underline{p}_L,\overline{p}_L]$ it holds that

$$(1-\alpha F_L(p))^{n-1}=\frac{\frac{\pi_L^*}{p}-(1-F_L(p))^{n-1}(1-\alpha)^n}{\alpha}\leq\frac{\pi_L^*}{\alpha p}. \tag{26}$$

For $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\overline{p}_L-(v_H-v_L-\frac{s}{\alpha})]=[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$, it holds that $p+(v_H-v_L-\frac{s}{\alpha})\in[\underline{p}_L,\overline{p}_L]$. Hence, by inequality (26), we have that for $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$,

$$\left[1-\alpha F_L\Big(p+\big(v_H-v_L-\frac{s}{\alpha}\big)\Big)\right]^{n-1}\leq\frac{\pi_L^*}{\alpha(p+v_H-v_L-\frac{s}{\alpha})}.$$

In turn, this implies that the following approximation can be given for equation (25) and $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$:

$$\begin{aligned}\pi_i(p)&=p\left[\alpha+(1-\alpha)\left[1-F_L\Big(p+v_H-v_L-\frac{s}{\alpha}\Big)\alpha\right]^{n-1}\right]\\&\leq p\left[\alpha+(1-\alpha)\left[\frac{\pi_L^*}{\alpha(p+v_H-v_L-\frac{s}{\alpha})}\right]\right].\end{aligned}$$

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p for $v_H-v_L-\frac{s}{\alpha}>0$ as assumed for the proposition, the last expression is thereby maximized for $p=\underline{p}_L$. This implies that for $p\in[\underline{p}_L-(v_H-v_L-\frac{s}{\alpha}),\underline{p}_L]$,[47]

$$\pi_i(p)\leq\underline{p}_L\left[\alpha+(1-\alpha)\left[\frac{\pi_L^*}{\alpha(\underline{p}_L+v_H-v_L-\frac{s}{\alpha})}\right]\right]=\pi_L^*.$$

Hence, deviations below $\underline{p}_L$ are indeed not profitable. This completes the proof. □

Proof of Proposition 4. In what follows, we prove existence. For uniqueness, the argument at the end of the proof of Proposition 2 applies.

[47] For $p<\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$, $\pi_i(p)<\pi_i(\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$, since all consumers already purchase deterministically at the deviating firm for $p=\underline{p}_L-(v_H-v_L-\frac{s}{\alpha})$.

Existence. The equilibrium objects $\underline{p}_M = \underline{p}_L$, $\overline{p}_M = \overline{p}_L$ and $\pi^*_M = \pi^*_L$ originate from solving the same system of equations (7) and (8) that define $\underline{p}_L$, $\overline{p}_L$ and π^*_L. Again, they are all strictly positive since $\alpha(v_H - v_L) - s > 0$ in the considered parameter region (compare also with the proof of Proposition 3). Observe next that $\underline{p}_M < v_L$ follows from $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} > \underline{\gamma}$, while $\overline{p}_M > v_L$ follows from $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} < \tilde{\gamma}$, as assumed for the proposition.

To see that $\underline{p}'_M > v_L$, note the following. First, since $v_L\left[\alpha(1-\alpha\kappa)^{n-1} + (1-\kappa)^{n-1}(1-\alpha)^n\right]$ is strictly increasing in v_L for $\kappa \in [0,1]$ while π^*_L is strictly decreasing in v_L, one can clearly see via the implicit definition of $\kappa = F_{M_1}(v_L)$ in equation (17) that κ must be strictly increasing in v_L whenever $\kappa \in [0,1)$. Moreover, for $\frac{v_L}{v_H} = \underline{\gamma}$ it holds that $\kappa = 0$, while for $\frac{v_L}{v_H} = \tilde{\gamma}$, it holds that $\kappa = 1$. Hence, $\kappa \in (0,1)$ in the considered parameter region. Substituting π^*_L from equation (17) into equation (16) now yields

$$\underline{p}'_M = v_L\left[1 + \frac{(1-\alpha)^n}{\alpha}\left(\frac{1-\kappa}{1-\alpha\kappa}\right)^{n-1}\right],$$

which indeed strictly exceeds v_L for all $\kappa \in [0,1)$.

A firm's expected profit when choosing a price in the range $[\underline{p}_M, v_L]$ is given by

$$\pi_i(p) = p\left[\alpha(1-\alpha F_{M_1}(p))^{n-1} + (1-F_{M_1}(p))^{n-1}(1-\alpha)^n\right],$$

such that for $F_{M_1}(p) = F_L(p)$, it clearly holds that $\pi_i(p) = \pi^*_L = \pi^*_M$ for all prices in that interval (as follows from the implicit definition of $F_L(p)$ in equation (11)). A firm's expected profit when choosing a price in the range$[\underline{p}'_M, \overline{p}_M]$ is given by $\pi_i(p) = p\left[1 - F_{M_2}(p)\alpha\right]^{n-1}\alpha$, such that for

$$F_{M_2}(p) = \frac{1}{\alpha}\left[1 - \left(\frac{\pi^*_L}{\alpha p}\right)^{\frac{1}{n-1}}\right],$$

it also holds that $\pi_i(p) = \pi^*_L = \pi^*_M$ for all prices in that interval. It is moreover easy to see that both $F_{M_1}(p)$ and $F_{M_2}(p)$ are strictly increasing in p. Hence, all equilibrium objects are well-behaved.

We now rule out profitable deviations outside the candidate equilibrium's pricing support. First, it clearly cannot be optimal to deviate to a price $p \in (v_L, \underline{p}'_M)$, as the deviating firm would not achieve a higher expected demand than when pricing at $\underline{p}'_M > p$. When deviating to a price $p > \overline{p}_M$, the deviating firm will only be searched if all rival firms price above $p - \left(v_H - v_L - \frac{s}{\alpha}\right)$ (compare with Lemma 1). Then, the deviating firm will cater to the mass $(1-\alpha)^{n-1}\alpha$ consumers who don't have a full match at any rival firm, but a full match at this firm. Thus, the expected profit at any such price $p > \overline{p}_M$ can be written as

$$\pi_i(p) = p\left[1 - F_{M_1}\left(p - \left(v_H - v_L - \frac{s}{\alpha}\right)\right)\right]^{n-1}(1-\alpha)^{n-1}\alpha, \tag{27}$$

where $F_{M_1}(p - (v_H - v_L - \frac{s}{\alpha}))$ is the relevant probability that a rival firm prices below $p - (v_H - v_L - \frac{s}{\alpha})$.[48] Since $F_{M_1}(\cdot) = F_L(\cdot)$, the same estimation as in the proof of Proposition 3 can now be used to show that $\pi_i(p) \leq \pi_L^* = \pi_M^*$ for all $p > \overline{p}_M$. Hence, deviations above $\overline{p}_M$ are not profitable.

We finally show that there are no profitable deviations to prices $p \in (0, \underline{p}_M)$. Following the argument in the proof of Proposition 3, a firm deviating to such a price makes an expected profit of

$$\pi_i(p) = p\left[\alpha + (1-\alpha)\left[1 - F_{M_r}\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right], \tag{28}$$

where $r = 1$ if $p + v_H - v_L - \frac{s}{\alpha} \leq v_L$ and $r = 2$ otherwise. Since $F_{M_1}(p)$ is implicitly defined by

$$p\left[\alpha(1-\alpha F_{M_1}(p))^{n-1} + (1-F_{M_1}(p))^{n-1}(1-\alpha)^n\right] - \pi_L^* = 0,$$

while $F_{M_2}(p)$ is implicitly defined by

$$p\left[\alpha(1-\alpha F_{M_2}(p))^{n-1}\right] - \pi_L^* = 0,$$

it is straightforward to see that $F_{M_1}(p) > F_{M_2}(p)$ when applied for the same price. Comparing with (28), a sufficient condition to have no profitable deviations below $\underline{p}_M$ is then that for all $p \in (0, \underline{p}_M)$,

$$\pi_i(p) \leq p\left[\alpha + (1-\alpha)\left[1 - F_{M_2}\left(p + v_H - v_L - \frac{s}{\alpha}\right)\alpha\right]^{n-1}\right] \overset{!}{\leq} \pi_L^*.$$

Inserting $F_{M_2}(\cdot)$ from equation (15), the above condition is equivalent to

$$p\left[\alpha + (1-\alpha)\left[\frac{\pi_L^*}{\alpha(p + v_H - v_L - \frac{s}{\alpha})}\right]\right] \overset{!}{\leq} \pi_L^* \qquad \forall p \in (0, \underline{p}_M).$$

[48] Otherwise, $p > v_H - \frac{s}{\alpha}$, implying zero demand for the deviating firm.

Since $\frac{p}{p+v_H-v_L-\frac{s}{\alpha}}$ is strictly increasing in p for $v_H - v_L - \frac{s}{\alpha} > 0$ as assumed for the proposition, the LHS in the last expression is maximized for $p = \underline{p}_M = \underline{p}_L$. Hence, for $p \in (0, \underline{p}_M]$,

$$\pi_i(p) \leq \underline{p}_M \left[\alpha + (1-\alpha)\left[\frac{\pi_L^*}{\alpha(\underline{p}_M + v_H - v_L - \frac{s}{\alpha})}\right]\right] = \pi_L^*,$$

such that deviations below $\underline{p}_M$ are indeed not profitable. This completes the proof. $\square$

Proof of Proposition 9. Note first that in the Bertrand-equilibrium region ($\alpha \leq \frac{s}{v_H}$, or $\alpha > \frac{s}{v_H}$ and $\frac{v_L}{v_H} \geq \overline{\gamma}$), the parameters are such that consumers will never search more than one firm. Given this, it is weakly optimal for a consumer to search the lowest-priced firm if it charges a weakly lower price than the expected gross utility it will provide, $p \leq \alpha v_H + (1-\alpha)v_L - s$ (compare with Lemma 1), where $\alpha v_H + (1-\alpha)v_L - s \leq v_L$ in the considered parameter region. The best firms can do is hence to coordinate their prices on $p^C = \alpha v_H + (1-\alpha)v_L - s \leq v_L$ for a maximal collusive stage-game profit of $\pi^C = \frac{\alpha v_H + (1-\alpha)v_L - s}{n}$.

If not in the Bertrand-equilibrium region, then there are only two price levels on which firms may optimally coordinate. First, for all symmetric collusive price levels $p^C \leq v_L$, all consumers will be served (randomly allocating themselves across firms in the search process), such that by pricing at v_L, each firm makes a maximal profit of $\pi^C = \frac{v_L}{n}$. Second, for all symmetric collusive price levels $p^C \in (v_L, v_H - \frac{s}{\alpha}]$, only consumers who find a full match at some firm will purchase eventually (again, randomly allocating themselves across firms in the search process).[49] The total number of such consumers is $1-(1-\alpha)^n$, for a maximal collusive per-firm profit of $\pi^C = \left(v_H - \frac{s}{\alpha}\right)\frac{1-(1-\alpha)^n}{n}$. Colluding on v_L is hence strictly more profitable if and only if

$$\frac{v_L}{v_H} > \left(1 - \frac{s}{\alpha v_H}\right)[1-(1-\alpha)^n] =: \gamma^C,$$

where it can be checked that $\gamma^C \in (\underline{\gamma}, \overline{\gamma})$ for $\alpha > \frac{s}{v_H}$,[50] as needs to hold (note that we have assumed that $\alpha < 1$ throughout).

[49] For $p^C > v_H - \frac{s}{\alpha}$, demand drops to zero.

[50] Given $\alpha > \frac{s}{v_H}$, the condition $\gamma^C > \underline{\gamma}$ is equivalent to

$$1 > (1-\alpha)^n + \alpha\left[\frac{(1-\alpha)^{n-1}}{(1-\alpha)^n + \alpha}\right],$$

which is true because $(1-\alpha)^n < 1-\alpha$ and $\frac{(1-\alpha)^{n-1}}{(1-\alpha)^n+\alpha} < 1$.

Observe finally that in each case, coordinating on the optimal collusive price level clearly gives rise to a higher stage-game profit π^C than the corresponding Nash-equilibrium profit π^N. Hence, the one-period gain when deviating optimally, π^D, will not be enough to make deviating worthwhile if firms are sufficiently patient (δ is sufficiently close to 1).[51]

$\square$

Proof of Proposition 10. The only statement which remains to be shown is that for any number of firms, there exists a parameter region where $\tilde{\gamma} > \gamma^C$. To see this, note first that

$$\tilde{\gamma}\Big(\alpha = \frac{s}{v_H}\Big) = \gamma^C\Big(\alpha = \frac{s}{v_H}\Big) = 0.$$

By continuity of $\tilde{\gamma}$ and γ^C, it thus suffices to establish that for any $n \geq 2$, we can find parameters such that

$$\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\bigg|_{\alpha=\frac{s}{v_H}} > \frac{d\gamma^C(\alpha)}{d\alpha}\bigg|_{\alpha=\frac{s}{v_H}}.$$

Now

$$\begin{aligned}\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\bigg|_{\alpha=\frac{s}{v_H}} &= \left(\frac{s}{\alpha^2 v_H}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right] + \Big(1-\frac{s}{\alpha v_H}\Big)\frac{d}{d\alpha}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right]\right)\bigg|_{\alpha=\frac{s}{v_H}} \\ &= \frac{1}{\alpha}\left[\frac{\alpha+(1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha}\right],\end{aligned}$$

where the first equality follows from direct calculation (compare with equation (9)) and the second equality follows from evaluating at $\alpha = \frac{s}{v_H}$. Likewise, we have that

$$\begin{aligned}\frac{d\gamma^C(\alpha)}{d\alpha}\bigg|_{\alpha=\frac{s}{v_H}} &= \left(\frac{s}{\alpha^2 v_H}[1-(1-\alpha)^n] + \Big(1-\frac{s}{\alpha v_H}\Big)\frac{d}{d\alpha}[1-(1-\alpha)^n]\right)\bigg|_{\alpha=\frac{s}{v_H}} \\ &= \frac{1}{\alpha}[1-(1-\alpha)^n],\end{aligned}$$

[51] As usual, the critical discount factor $\overline{\delta}$ above which the optimal collusive scheme can be supported via grim-trigger strategies is given by $\overline{\delta} = \frac{\pi^D-\pi^C}{\pi^D-\pi^N}$. For each parameter constellation, π^N is known from Propositions 2 to 4, while π^C is known from the above analysis. It is moreover easy to see that in the high-price equilibrium region ($p^C = v_H - \frac{s}{\alpha}$) and Bertrand-equilibrium region ($p^C = \alpha v_H + (1-\alpha)v_L - s$), the respective optimal deviation from p^C is always to undercut marginally, giving rise to a single optimal deviation profit and thereby a single critical discount factor in either region. However, in the gap equilibrium region and low-price equilibrium region, which share the same π^N, either $p^C = v_H - \frac{s}{\alpha}$ or $p^C = v_L$ can constitute the optimal collusive price level, depending on v_L/v_H. Moreover, it can be shown that in both cases, the optimal deviation can be to undercut marginally *or* to undercut substantially to stop all consumers from searching on. This implies that four different critical discount factors emerge in these regions. In total, one of six critical discount factors is thus the relevant one for supporting the optimal collusive scheme via grim-trigger strategies. Since this is not the main interest of this article, the various critical discount factors are not reported here. Further details are available from the authors upon request.

where the first equality follows from direct calculation (compare with equation (21)) and the second equality follows from evaluating at $\alpha = \frac{s}{v_H}$. Hence, it holds that $\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}} > \frac{d\gamma^C(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}}$ if and only if

$$h_1(\alpha,n) := \frac{\alpha + (1-\alpha)^n}{(1-\alpha)^{n-1}(2-3\alpha)+2\alpha} > 1-(1-\alpha)^n := h_2(\alpha,n).$$

Observe next that $h_1(0,n) = \frac{1}{2} > 0 = h_2(0,n)$. Hence, for $\alpha = 0$, it holds that $h_1(\alpha,n) > h_2(\alpha,n)$, such that by continuity of h_1 and h_2 in α, the equality remains true also for slightly positive α, say up to $\alpha_{max}(n) > 0$. When now s is sufficiently small with $s \in [0, \alpha_{max}(n)v_H]$ such that $\frac{s}{v_H} \leq \alpha_{max}(n)$, the inequality $\frac{d\tilde{\gamma}(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}} > \frac{d\gamma^C(\alpha)}{d\alpha}\Big|_{\alpha=\frac{s}{v_H}}$ is thus satisfied. This completes the proof. □

Proof of Proposition 11. As argued in the main text, from the cartel's perspective it can only be optimal that all firms choose the highest possible price which induces search, $p_i^K = v_H - \frac{s}{\alpha}$ for all $i \in \{1,...,n\}$, or that exactly one firm sets the low price v_L while all other $n-1$ firms set the highest possible price $v_H - \frac{s}{\alpha}$ (compare also with Footnote 41). The former implies a cartel profit of $\pi^K = \left(v_H - \frac{s}{\alpha}\right)[1-(1-\alpha)^n]$, the latter a cartel profit of

$$\pi^K = v_L[\alpha + (1-\alpha)^n] + \left(v_H - \frac{s}{\alpha}\right)(1-\alpha)\left[1-(1-\alpha)^{n-1}\right].$$

Comparing these two, it turns out that the former is strictly better if and only if

$$\frac{v_L}{v_H} < \left(1 - \frac{s}{\alpha v_H}\right)\frac{\alpha}{\alpha + (1-\alpha)^n} =: \gamma^K,$$

as reported in the proposition. That $\gamma^K > \underline{\gamma}$ is trivial to check, whereas $\gamma^K < \gamma^C$ has already been shown in Footnote 42. The final welfare statements are obvious when comparing the different regimes. □

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Price-Directed Search and Collusion

Abstract
In many (online) markets, consumers can readily observe prices, but need to examine individual products at positive cost in order to assess how well they match their needs. We propose a tractable model of price-directed sequential search in a market where firms compete in prices. Each product meets consumers' basic needs, however they are only fully satisfied with a certain probability. In our setup, four types of pricing equilibria emerge, some of which entail inefficiencies as not all consumers are (always) served. We then lend our model to analyze collusion. We find that for any number of firms, there exists a parameter region in which the payoff-dominant symmetric collusive equilibrium gives rise to a higher expected total social welfare than the repeated one-shot Nash equilibrium. In other regions, welfare is identical under collusion and merely consumer rents are transferred, or both welfare and consumer rents are reduced. An all-inclusive cartel maximizing industry profit increases welfare for an even larger set of parameters, but may also be detrimental to it.

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