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Liapounoff's vector measure theorem in Banach spaces

Michael Greinecker* and Konrad Podczeck†

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Abstract

We present a result on convexity and weak compactness of the range of a vector measure with values in a Banach space, based on the Maharam classification of measure spaces. Our result extends a recent result of Khan and Sagara [Illinois Journal of Mathematics, forthcoming].

Subject classification: 28B05, 46G10

Keywords: Liapounoff's theorem, vector measures, multifunctions

1 Introduction. In various contexts, it was observed that measure spaces of uncountable Maharam type allow for useful constructions that are not available for the unit interval with Lebesgue measure; see, e.g., Scott [1967], Hoover and Keisler [1984], Rustichini and Yannelis [1991], Podczeck [2008], or Keisler and Sun [2009]. Continuing this line, Khan and Sagara [2013] recently established a version of Liapounoff's theorem for vector measures with values in a Banach space.

The main result in Khan and Sagara [2013] says that if (T, Σ, μ) is a totally finite measure space and G is a μ-continuous countably additive vector measure defined on Σ with values in a Banach space X, then the range of G is a convex and weakly compact set in X provided that μ is Maharam-type-homogeneous with Maharam type strictly larger then the algebraic dimension of X.

In this note, we sharpen this result by Khan and Sagara [2013]. In particular, we remove the hypothesis of Maharam type homogeneity, and, in the condition on the codomain of a vector measure, replace the algebraic dimension of a Banach space by the cardinal of some point-separating family of continuous linear functionals. The latter has drastic consequences; see the comments after Corollary 5 below.

We provide two proofs of our main result. Our first proof is very short. It reveals that versions of Liapounoff's theorem based on the Maharam classification of measure spaces are, in fact, a straightforward consequence of Knowles'

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version of Liapounoff's theorem in the weak topology [Knowles, 1975], which we use in the form as stated in Diestel and Uhl [1977, Theorem IX.1.4].

Our second proof is essentially measure-theoretic and works without the extreme point arguments on which the proof of Diestel and Uhl [1977, Theorem IX.1.4] is based. Instead, it makes use of an idea introduced by Maharam [1942] in the proof of her classification result for measure algebras.

We illustrate the usefulness of our results by showing how they allow to prove the convexity of the Aumann integral of a multifunction taking values in a Banach space, following the lines of the classical proof by Richter [1963] for the case of a finite-dimensional codomain.

2 Notation and terminology. If A is a set, $\#(A)$ denotes its cardinal. As usual, ω denotes the least infinite cardinal, and ω_1 the least uncountable cardinal.

If X is a Banach space, X^* denotes its dual space, and for a set $U \subseteq X$, $\mathrm{dens}(U)$ denotes the density of U, i.e., the least cardinal of any set $V \subseteq U$ which is dense in U.

Let (T, Σ, μ) be a totally finite measure space. If $E \subseteq T$, μ_E denotes the subspace measure on E defined from μ. By $\mathfrak{A}_\mu$ we denote the measure algebra of μ, and for $a \in \mathfrak{A}_\mu$, by $\mathfrak{A}_a$ the principal ideal in $\mathfrak{A}_\mu$ generated by a. If κ is an infinite cardinal, we say that (T, Σ, μ), or the measure μ, is κ-atomless if for each non-zero $a \in \mathfrak{A}_\mu$, $\mathfrak{A}_a$ has Maharam type at least κ. (Recall that the Maharam type of a Boolean algebra $\mathfrak{A}$ is the least cardinal of a set $\mathcal{H} \subseteq \mathfrak{A}$ such that the order-closed sub-algebra of $\mathfrak{A}$ generated by $\mathcal{H}$ is $\mathfrak{A}$ itself, and that any principal ideal of a Boolean algebra can be viewed as a Boolean algebra in its own right.) Note that "ω-atomless" means just "atomless" in the usual sense. Note also that being "κ-atomless" does not imply being Maharam-type-homogeneous.

We shall make use of the following facts.

Fact 1. *Let (T, Σ, μ) be a totally finite measure space, and κ an infinite cardinal. Then μ is κ-atomless if and only if $\mathrm{dens}(L_1(\mu_E)) \geqslant \kappa$ for each $E \in \Sigma$ with $\mu(E) > 0$.*

(Apply Fremlin [2004, 331Y(e) or 365Y] to the non-zero principal ideals of the measure algebra of (T, Σ, μ).)

Fact 2. *If X is an infinite-dimensional Banach space, then $\mathrm{dens}(X)$ is equal to the least cardinal of a set $A \subseteq X$ such that A separates the points of X^*.*

(If $A \subseteq X$ is dense then A separates the points of X^*. Conversely, let $A \subseteq X$ separate the points of X^*. Then $\overline{\mathrm{span}}\, A = X$ by the Hahn-Banach theorem; in particular, $\#(A)$ must be infinite if X is infinite-dimensional. Now, writing F for the set of all (finite) linear combinations with rational coefficients of members of A, F is dense in $\overline{\mathrm{span}}\, A$, and $\#(F) = \#(A)$ if $\#(A)$ is infinite.)

Finally, if (T, Σ, μ) is a totally finite measure space, X a Banach space, and $G\colon \Sigma \to X$ a countably additive μ-continuous vector measure, then for any $f \in L_\infty(\mu)$, $\int f \, dG$ denotes the Bartle integral; see Diestel and Uhl [1977, pp. 5 and pp. 56].

3 The vector measure theorem. Here is our version of Liapounoff's theorem.

Theorem. *Let* (T, Σ, μ) *be a totally finite measure space,* X *a Banach space, and* $G\colon \Sigma \to X$ *a* μ*-continuous countably additive vector measure. Let* κ *be an infinite cardinal and assume that* μ *is* κ*-atomless and that there is a family* $\langle x_i^* \rangle_{i \in I}$ *in* X^**, with* $\#(I) < \kappa$*, which separates the points of* $\overline{\operatorname{span}}\, G(\Sigma)$*. Then for every* $E \in \Sigma$*,* $\{G(A \cap E) \colon A \in \Sigma\}$ *is a weakly compact and convex set in* X*.*

Proof. By Diestel and Uhl [1977, Theorem IX.1.4], we need to show that for any $E \in \Sigma$ with $\mu(E) > 0$, the operator $T_E \colon L_\infty(\mu_E) \to X$ given by $T_E(f) = \int_E f \, dG_E$ is not an injection, where G_E is the restriction of G to $\{A \cap E \colon A \in \Sigma\}$.[1] Fix any such E. By Diestel and Uhl [1977, Lemma IX.1.3], T_E is weak*-weakly continuous, so for each $i \in I$, $x_i^* T_E$ is a weak*-continuous linear functional on $L_\infty(\mu_E)$ and can be identified with an element of $L_1(\mu_E)$. Note also that T_E takes all of its values in $\overline{\operatorname{span}}\, G(\Sigma)$ [use Diestel and Uhl, 1977, Lemma IX.1.3(c)].

The hypothesis on μ implies that $L_1(\mu_E)$ is infinite-dimensional. Moreover by Fact 1, $\operatorname{dens}(L_1(\mu_E)) \geqslant \kappa$. Thus, by Fact 2, the family $\langle x_i^* T_E \rangle_{i \in I}$ in $L_1(\mu_E)$ cannot separate the points of $L_\infty(\mu_E)$, as $\#(I) < \kappa$, so T_E is not an injection, as T_E takes values in $\overline{\operatorname{span}}\, G(\Sigma)$ and $\langle x_i^* \rangle_{i \in I}$ separates the points of $\overline{\operatorname{span}}\, G(\Sigma)$. □

The theorem yields several corollaries, with the intended interpretation of X being infinite-dimensional. For the first corollary, just recall that the density of any Banach space X is at least as large as the least cardinal of any $A \subseteq X^*$ such that A separates the points of X [cf. Fabian et al., 2001, page 358].

Corollary 1. *Let* (T, Σ, μ) *be a totally finite measure space, and* X *a Banach space. Assume that for some uncountable cardinal* κ*,* μ *is* κ*-atomless with* $\kappa > \operatorname{dens}(X)$*. Then for any* μ*-continuous countably additive vector measure* $G\colon \Sigma \to X$*,* $\{G(A \cap E) \colon A \in \Sigma\}$ *is a weakly compact and convex set in* X *for every* $E \in \Sigma$*.*

The next corollary of our theorem is more general; see Remark 1 below.

Corollary 2. *Let* (T, Σ, μ) *be a totally finite measure space,* X *a Banach space, and* $G\colon \Sigma \to X$ *a* μ*-continuous countably additive vector measure. Assume that for some uncountable cardinal* κ*,* μ *is* κ*-atomless with* $\kappa > \operatorname{dens}(G(\Sigma)$*. Then for every* $E \in \Sigma$*,* $\{G(A \cap E) \colon A \in \Sigma\}$ *is a weakly compact and convex set in* X*.*

Proof. The linear combinations with rational coefficients of the members of a dense subset of $G(\Sigma)$ are dense in $\overline{\operatorname{span}}\, G(\Sigma)$, so the hypothesis implies that $\kappa > \operatorname{dens}(\overline{\operatorname{span}}\, G(\Sigma))$. Apply Corollary 1 to $\overline{\operatorname{span}}\, G(\Sigma)$. □

The following special case of Corollary 1 is the content of Theorem 4.1 in Khan and Sagara [2013].

Corollary 3. *Let* (T, Σ, μ) *be a totally finite measure space, and* X *a separable Banach space. If* μ *is* ω_1*-atomless, then for any* μ*-continuous countably additive vector measure* $G\colon \Sigma \to X$*,* $\{G(A \cap E) \colon A \in \Sigma\}$ *is a weakly compact and convex set in* X *for every* $E \in \Sigma$*.*

[1] Note for this reference that $L_\infty(\mu_E)$ can be identified with the subspace of $L_\infty(\mu)$ consisting of the elements vanishing of E.

If X is a dual Banach space, say $X = Y^*$, then any dense subset of Y separates the points of X. Thus the above theorem also implies the following result.

Corollary 4. *Let (T, Σ, μ) be a totally finite measure space, and X a dual Banach space, say $X = Y^*$. Assume that for some uncountable cardinal κ, μ is κ-atomless with $\kappa > \operatorname{dens}(Y)$. Then for any μ-continuous countably additive vector measure $G\colon \Sigma \to X$, $\{G(A \cap E)\colon A \in \Sigma\}$ is a weakly compact and convex set in X for every $E \in \Sigma$.*

A particular case of Corollary 4 is noted next.

Corollary 5. *Let (T, Σ, μ) be a totally finite measure space, and X the dual of a separable Banach space. If μ is ω_1-atomless, then for any μ-continuous countably additive vector measure $G\colon \Sigma \to X$, $\{G(A \cap E)\colon A \in \Sigma\}$ is a weakly compact and convex set in X for every $E \in \Sigma$.*

Remark 1. The condition in Corollary 2 is more general than that in Corollary 1. In fact, the range of a Banach space valued countably additive vector measure defined on a σ-algebra is always relatively weakly compact [Diestel and Uhl, 1977, Corollary I.2.7], and there are plenty of non-separable Banach spaces in which every weakly compact subset is (norm) separable. E.g., weakly compact subsets of $C(K)$—the space of continuous real-valued functions on a compact Hausdorff space K, endowed with its usual norm—are (norm) separable whenever K carries a Radon measure with full support [Rosenthal, 1969, Theorem 1.4].

Remark 2. Corollary 1 considerably improves Theorem 5.1 in Khan and Sagara [2013] where the measure space domain (T, Σ, μ) is required to be Maharam-type-homogeneous, and the Maharam-type of μ to be strictly larger then the algebraic dimension of the codomain of a vector measure. Note that any infinite cardinal is possible as the density of a Banach space; e.g., $\operatorname{dens}(\ell_2(\kappa)) = \kappa$ if κ is any infinite cardinal. On the other hand, the algebraic dimension of an infinite-dimensional Banach space is at least $\mathfrak{c} = 2^\omega$ [Mackey, 1945, Theorem I-1], and by Easton's theorem [see Easton, 1970], the only restriction the ZFC axioms put on the cardinal of 2^ω is that it has uncountable cofinality and does not exceed the cardinal of 2^{ω_1}.

Remark 3. Another improvement of our results over those in Khan and Sagara [2013] is provided by Corollary 5. E.g., let $X = M[0, 1]$, the space of bounded signed Borel measures on $[0, 1]$ with the total variation norm. Then X is non-separable, but $X = Y^*$ where $Y = C[0, 1]$, the space of continuous functions on $[0, 1]$ with the sup-norm, which is a separable space. For the conclusion of Corollary 5, the results in Khan and Sagara [2013] would require κ to be strictly larger than the algebraic dimension of $M[0, 1]$, i.e., $\kappa > 2^\omega$, while, as shown by our Corollary 5, $\kappa = \omega_1$ suffices.

Remark 4. For any infinite cardinal κ, the "$<$" in our theorem cannot be replaced by "$\leqslant$". This may be seen by transforming an example in Uhl [1969] as follows. Fix any infinite cardinal κ. Let μ be the usual measure on $\{0,1\}^\kappa$, and Σ its domain. Define $G\colon \Sigma \to L_1(\mu)$ by setting $G(E) = 1_E$ for each $E \in \Sigma$. Then G is a μ-continuous countably additive vector measure such that $G(\Sigma)$ is not a convex set in $L_1(\mu)$. Also, $\overline{\operatorname{span}}\, G(\Sigma) = L_1(\mu)$. Now by the choice of μ, μ is Maharam-type-homogeneous with Maharam type κ; thus μ is κ-atomless in the terminology of this note. In particular, $L_1(\mu)$ is infinite-dimensional. Moreover, $\operatorname{dens}(L_1(\mu)) = \kappa$. By [Fabian et al., 2001, Example (v), page 358], the Banach space $L_1(\mu)$ is weakly compactly generated, so by [Fabian et al., 2001, Theorem 11.3], the weak*-density of its dual is also κ. As $L_1(\mu)$ is infinite-dimensional, the least cardinal of any point-separating family of continuous linear functionals on $L_1(\mu)$ is κ, too.[2]

Remark 5. As shown in Lemma 4 in Podczeck [2008],[3] if X is any infinite-dimensional Banach space and (T, Σ, μ) is any totally finite measure space such that μ is not ω_1-atomless, then there is a Bochner-integrable function $f\colon T \to X$ such that the set $\left\{\int_E f\,d\mu\colon E \in \Sigma\right\}$ is not a convex subset of X.[4] Translating this into vector measures terms yields directly the following fact [see also Khan and Sagara, 2013, Lemma 4.1 and Theorem 4.2(ii)⇒(i)].

Proposition 1. *Let (T, Σ, μ) be a totally finite measure space, and X an infinite-dimensional Banach space. If μ is not ω_1-atomless, then there is μ-continuous countably additive vector measure $G\colon \Sigma \to X$ such that the set $\{G(A)\colon A \in \Sigma\}$ is not a convex subset of X.*

Combining this with Corollary 3 we obtain the following statement.

Proposition 2. *Let (T, Σ, μ) be a totally finite measure space, and X a separable infinite-dimensional Banach space. In order that for any μ-continuous countably additive vector measure $G\colon \Sigma \to X$, the set $\{G(A \cap E)\colon A \in \Sigma\}$ is a weakly compact and convex subset of X for every $E \in \Sigma$, it is both necessary and sufficient that μ is ω_1-atomless.*

Remark 6. By what was said in Remark 1, an analog of the necessity part of Proposition 2 formulated by replacing, for an uncountable cardinal κ, "separable infinite-dimensional" in the condition on X by $\operatorname{dens}(X) = \kappa$, and ω_1-atomless by κ^+-atomless, where κ^+ is the cardinal successor of κ, is wrong.

4 An alternative proof of the theorem for κ uncountable. If the range of a vector measure is not finite-dimensional, then our theorem requires the cardinal κ to be uncountable. For such a κ, we can present an alternative proof of our theorem. It is important to note that this alternative proof does not apply

[2]The weak*-density of the dual X^* of any infinite-dimensional Banach space X is equal to the least cardinal of a subset of X^* separating the points of X, which follows similarly as Fact 2 above.

[3]The construction in the proof of that lemma follows that in the proof of Diestel and Uhl [1977, Corollary IX.1.6]

[4]Recall the standard fact that if (T, Σ, μ) is a totally finite measure space, X a Banach space, and $f\colon T \to X$ is Bochner integrable, then the indefinite Bochner integral of f is a μ-continuous countably additive vector measure.

to the classical situation of a finite-dimensional vector measure which is just assumed to be atomless; see Remark 8 below. On the other hand, the proof we will present in this section does not rely on Liapounoff's theorem in the weak topology [Diestel and Uhl, 1977, Theorem IX.1.4], and in particular, is not based on non-injectivity of certain linear operators. Altogether, this reveals that, in the terminology of the statement of our theorem, $\kappa > \#(I)$ with κ uncountable but $\#(I)$ allowed to be infinite is not simply an analog of $\omega > \#(I)$—the situation of the classical Liapounoff theorem—on a higher level of cardinals.

For convenience of reference, we start by recalling three measure theoretic facts. The first two may be deduced from Fremlin [2004, 321X(b) and 313M(b), respectively].

Fact 3. *Let* (T, Σ, μ) *be a totally finite measure space, and* Σ_1 *a sub-σ-algebra of* Σ*. If* Σ_1 *is generated by a set* $\mathcal{A} \subseteq \Sigma_1$*, then the order-closed sub-algebra of* $\mathfrak{A}_\mu$ *generated by the* μ*-equivalence classes of the members of* $\mathcal{A}$ *is the same as the set of the* μ*-equivalences of the members of* Σ_1*.*

Fact 4. *Let* $\mathfrak{A}$ *be a Boolean algebra, let* $\mathcal{H} \subseteq \mathfrak{A}$*, and let* $\mathfrak{B}$ *be the order-closed sub-algebra of* $\mathfrak{A}$ *generated by* $\mathcal{H}$*. Then for each* $a \in \mathfrak{A}$*,* $\{b \cap a \colon b \in \mathfrak{B}\}$ *includes the order-closed sub-algebra of* $\mathfrak{A}_a$ *generated by* $\{h \cap a \colon h \in \mathcal{H}\}$ *(treating* $\mathfrak{A}_a$ *as a Boolean algebra in its own right).*

For the next fact, see Fremlin [2004, 331B]. This fact provided the basis on which our alternative method of proof relies. Recall that if $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ an order-closed sub-algebra of $\mathfrak{A}$, then $\mathfrak{A}$ is called *relatively atomless* over $\mathfrak{B}$ if $\{a \cap b \colon b \in \mathfrak{B}\} \neq \mathfrak{A}_a$ for each non-zero $a \in \mathfrak{A}$ (where $\mathfrak{A}_a$ is the principal ideal in $\mathfrak{A}$ generated by a and where "$\cap$" has the usual meaning in the context of Boolean algebras).

Fact 5. *Let* $(\mathfrak{A}, \bar{\mu})$ *be a totally finite measure algebra and* $\mathfrak{B}$ *an order-closed sub-algebra of* $\mathfrak{A}$ *such that* $\mathfrak{A}$ *is relatively atomless over* $\mathfrak{B}$*. Let* $\bar{\nu} \colon \mathfrak{B} \to \mathbb{R}$ *be an additive functional such that* $0 \leqslant \bar{\nu}(b) \leqslant \bar{\mu}(b)$ *for every* $b \in \mathfrak{B}$*. Then there is a* $c \in \mathfrak{A}$ *such that* $\bar{\nu}(b) = \bar{\mu}(b \cap c)$ *for every* $b \in \mathfrak{B}$*.*

Lemma 1. *Let* κ *be an infinite cardinal,* (T, Σ, μ) *a* κ*-atomless totally finite measure space, and* Σ_1 *a sub-σ-algebra of* Σ *generated by a family* $\mathcal{A} \subseteq \Sigma_1$ *with* $\#(\mathcal{A}) < \kappa$*. Then given any* Σ*-measurable* $f \colon T \to [0, 1]$ *there is an* $F \in \Sigma$ *such that* $\int_H f \, d\mu = \mu(H \cap F)$ *for each* $H \in \Sigma_1$*.*

Proof. Write $\mathfrak{B}$ for the order-closed sub-algebra of $\mathfrak{A}_\mu$ generated by the (μ-equivalence classes of the) members of $\mathcal{A}$. In view of Fact 4, the hypotheses imply that for each non-zero $a \in \mathfrak{A}_\mu$ we have $\{a \cap b \colon b \in \mathfrak{B}\} \neq \mathfrak{A}_a$, i.e., $\mathfrak{A}_\mu$ is relatively atomless over $\mathfrak{B}$. For each $b \in \mathfrak{B}$ let H_b be a version of b and define $\bar{\nu} \colon \mathfrak{B} \to \mathbb{R}$ by setting $\bar{\nu}(b) = \int_{H_b} f \, d\mu$ for $b \in \mathfrak{B}$. Note that $\bar{\nu}$ is additive with $\bar{\nu}(b) \leqslant \bar{\mu}(b)$ for each $b \in \mathfrak{B}$ (where $\bar{\mu}$ is the measure algebra functional defined from μ). The claim now follows from Fact 5 in conjunction with Fact 3. □

Lemma 2. *Let κ be an infinite cardinal, (T, Σ, μ) a κ-atomless totally finite measure space, $f\colon T \to [0,1]$ a Σ-measurable function, and Σ_1 a sub-σ-algebra of Σ generated by a family $\mathcal{A} \subseteq \Sigma_1$ with $\#(\mathcal{A}) < \kappa$. Then there is an $F \in \Sigma$ such that $\int_T hf \, d\mu = \int_F h \, d\mu$ for each integrable and Σ_1-measurable $h\colon T \to \mathbb{R}$.*

Proof. Let F be chosen according to Lemma 1, so that $\int_T hf \, d\mu = \int_F h \, d\mu$ holds whenever h is the characteristic function of some $H \in \Sigma_1$. It follows that this equality also holds for h being any Σ_1-measurable simple function, and thus for h being any μ-integrable and Σ_1-measurable function. □

Proof of the theorem for κ uncountable. We need only consider the case $E = T$. (Let $E \in \Sigma$. If $\mu(E) = 0$, G vanishes on E. If $\mu(E) > 0$, then μ_E is κ-atomless, for the same κ for which μ is, directly by the definition of this property; further, $A \subseteq B \subseteq X$ implies $\operatorname{dens}(A) \leqslant \operatorname{dens}(B)$.) Now by Diestel and Uhl [1977, Lemma IX.1.3],

$$\overline{\text{co}}\, G(\Sigma) = \left\{ \int_T f \, dG \colon 0 \leqslant f \leqslant 1, f \in L_\infty(\mu) \right\},^{5}$$

and by Diestel and Uhl [1977, Corollary I.2.7], $G(\Sigma)$ is relatively weakly compact, so $\overline{\text{co}}\, G(\Sigma)$ is weakly compact by the Krein-Smulian theorem. Thus it suffices to show that given any $f \in L_\infty(\mu)$ with $0 \leqslant f \leqslant 1$ there is an $F \in \Sigma$ with $G(F) = \int_T f \, dG$. Pick any such f.

Invoking the family $\langle x_i^* \rangle_{i \in I}$ hypothesized, note that for each $i \in I$, $x_i^* G$ is a real-valued μ-continuous signed measure, and therefore has a Radon-Nikodym derivative $h_i \in L_1(\mu)$ which we identify with one of its versions. Now each h_i is measurable for some countably generated sub-σ-algebra of Σ. Noting that $\#(I) \cdot \omega < \kappa$, because κ is assumed to be uncountable, we can therefore find a sub-σ-algebra Σ_1 of Σ such that each h_i is Σ_1-measurable and such that Σ_1 is generated by a family $\mathcal{A} \subseteq \Sigma_1$ with $\#(\mathcal{A}) < \kappa$. By Lemma 2, we can find an $F \in \Sigma$ such that $\int_T h_i f \, d\mu = \int_F h_i \, d\mu$ for each $i \in I$. Now for each $i \in I$ we have

$$x_i^* \int_T f \, dG = \int_T f \, dx_i^* G = \int_T h_i f \, d\mu = \int_F h_i \, d\mu = x_i^* G(F),$$

showing that $G(F) = \int_T f \, dG$, as $\langle x_i^* \rangle_{i \in I}$ separates the points of $\overline{\text{co}}\, G(\Sigma)$. □

Remark 7. The heart of the proof as given in this section is Lemma 1. It depends on Fact 5, which originated as part of the proof in Maharam [1942]. Exploiting the fact stated in this lemma is what allows to bypass the usual extreme point arguments in proofs of Liapounoff's theorem.

Remark 8. As announced above, the method of the proof given in this section does not capture the classical situation where the Banach space X is finite-dimensional and (T, Σ, μ) is just assumed to be atomless, i.e., ω-atomless in

[5] For this equality to be valid, the full hypothesis in Diestel and Uhl [1977, Lemma IX.1.3] that $G(E \cap F) = 0$ if and only if $\mu(E) = 0$ is not needed; the part requiring μ-continuity of G suffices, as may be seen form the proofs of Diestel and Uhl [1977, Lemma IX.1.3 and Corollary I.2.7].

the terminology of this note. To see this, note that for $\kappa = \omega$ the inequality $\#(I) \cdot \omega < \kappa$ is of course wrong (unless $\#(I) = 0$). But this strict inequality is necessary in order to apply Lemma 1 and, a fortiori, Lemma 2.

In order to apply the alternative method proof to the classical case, one would therefore need to replace the strict inequality in Lemma 1 by a weak inequality. But this cannot be done. For let $T = [0, 1]$, $\Sigma = \Sigma_1$ the Borel σ-algebra of $[0, 1]$, μ Lebesgue measure restricted to Σ, and f the constant function taking value $1/2$. If the conclusion of Lemma 1 would hold in this case, there would be an $F \in \Sigma$ such that $\int_H f \, d\mu = \mu(H)/2 = \mu(H \cap F)$ for all $H \in \Sigma$. But then we would have $\mu(F) = 1/2$, and on the other hand, setting $H = F$, $\mu(F) = \mu(F)/2$, which is absurd.

5 Application. Applying the above results may require an intermediate step of the following kind.

Lemma 3. *Let* (T, Σ, μ) *be a totally finite measure space,* X *a Banach space, and* $G, F \colon \Sigma \to X$ *two* μ*-continuous countably additive vector measures. Let* $E \in \Sigma$ *and* $0 < \alpha < 1$*. Then there is an* $H \subseteq \Sigma$ *with* $H \subseteq E$ *such that* $\alpha G(E) + (1 - \alpha)F(E) = G(H) + F(E \backslash H)$ *if any of the following conditions hold.*

(a) μ *is* ω_1*-atomless and there are separable closed linear subspaces* S_1 *and* S_2 *of* X *such that* $G(\Sigma) \subseteq S_1$ *and* $F(\Sigma) \subseteq S_2$*.*

(b) For some uncountable cardinal κ*,* μ *is* κ*-atomless with* $\kappa > \text{dens}(X)$*.*

(c) X *is a dual space, say* $X = Y^*$*, and for some uncountable cardinal* κ*,* μ *is* κ*-atomless with* $\kappa > \text{dens}(Y)$*.*

Proof. The parallel product $(G, F) \colon \Sigma \to X \times X$, i.e., $(G, F)(A) = (G(A), F(A))$ for all $A \in \Sigma$, is again a μ-continuous countably additive vector measure. If (a) holds then (F, G) can be viewed as a vector measure taking values in the separable Banach space $S_1 \times S_2$. For (b), note that $\text{dens}(X \times X) = \text{dens}(X)$, and for (c) note that $\text{dens}(Y \times Y) = \text{dens}(Y)$ and that $(Y \times Y)^*$ may be identified with $X \times X$. Thus, under any of the three conditions, by Corollaries 3, 1, and 4, respectively, there is an $H \in \Sigma$ with $H \subseteq E$ such that $\alpha(G(E), F(E)) = (G(H), F(H))$. Now $F(E \backslash H) = F(E) - F(H) = (1 - \alpha)F(E)$, so $G(H) + F(E \backslash H) = \alpha G(E) + (1 - \alpha)F(E)$. □

Integrals of multifunctions. Let (T, Σ, μ) be a totally finite measure space, X a Banach space, and $\varphi \colon T \to 2^X$ a multifunction. The set of Bochner integrals of all Bochner integrable selectors of φ is the *Aumann-Bochner integral of* φ, the set of Pettis integrals of all Pettis integrable selectors of φ is the *Aumann-Pettis integral of* φ, and if X is a dual Banach space, then the set of Gelfand integrals of all Gelfand integrable selectors of φ is the *Aumann-Gelfand integral of* φ.

The multifunction φ is called *integrably bounded*, if for some integrable function $\rho \colon T \to \mathbb{R}_+$, $\sup\{\|x\| \colon x \in \varphi(t)\} \leqslant \rho(t)$ for almost all $t \in T$.

To deal with the Gelfand integral of a multifunction, we need some preparation.

Lemma 4. *Let* (T, Σ, μ) *be a totally finite measure space,* X *a dual Banach space, and* $f\colon T \to X$ *a Gelfand integrable function. Suppose that for some integrable function* $\rho\colon T \to \mathbb{R}_+$*,* $\|f(t)\| \leqslant \rho(t)$ *for almost all* $t \in T$*. Then the indefinite Gelfand integral of* f *is a* μ*-continuous countably additive vector measure.*

Proof. Write $G\colon \Sigma \to X$ for the indefinite Gelfand integral of f. Thus for each $E \in \Sigma$, $G(E)$ is the Gelfand integral of f over E. Clearly G is an additive vector measure. Suppose $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in Σ with $\mu(E_n) \to 0$. By hypothesis, we have $X = Y^*$ for some Banach space Y; write B_Y for the closed unit ball in Y. Note that for any $y \in B_Y$, we have $|yf(t)| \leqslant \|f(t)\| \leqslant \rho(t)$ for almost all $t \in T$. Thus, using the definition of the Gelfand integral,

$$\begin{aligned}\|g(E_n)\| &= \sup\{|yG(E_n)| \colon y \in B_Y\} = \sup\left\{\left|\int_{E_n} yf(t)\, d\mu(t)\right| \colon y \in B_Y\right\} \\ &\leqslant \sup\left\{\int_{E_n} |yf(t)|\, d\mu(t) \colon y \in B_Y\right\} \leqslant \int_{E_n} \rho(t)\, d\mu(t) \to 0\end{aligned}$$

as $n \to \infty$. Thus G is μ-continuous, therefore also countably additive. □

Proposition 3. *Let* (T, Σ, μ) *be a totally finite measure space,* X *a Banach space, and* $\varphi\colon T \to 2^X$ *a multifunction.*

(a) If μ *is* ω_1*-atomless then the Aumann-Bochner integral of* φ *is convex.*

(b) If for some uncountable cardinal κ*,* μ *is* κ*-atomless with* $\kappa > \operatorname{dens}(X)$*, then the Aumann-Pettis integral of* φ *is convex.*

(c) Suppose X *is a dual space, say* $X = Y^*$*, and that* φ *is integrably bounded. If for some uncountable cardinal* κ*,* μ *is* κ*-atomless with* $\kappa > \operatorname{dens}(Y)$*, then the Aumann-Gelfand integral of* φ *is convex.*

Proof. The indefinite Pettis integral of a Pettis integrable $f\colon T \to X$ and, in particular, the indefinite Bochner integral of a Bochner integrable $f\colon T \to X$, are μ-continuous countably additive vector measures [Diestel and Uhl, 1977, Theorem II.3.5]. Under (c), Lemma 4 shows that the same is true for the indefinite Gelfand integral of a Gelfand integrable selector $f\colon T \to X$ of φ. Note also that the range of the indefinite Bochner integral of a Bochner integrable function $f\colon T \to X$ is included in a separable subspace of X.

The proposition now follows from Lemma 3, with E there replaced by T, noting that if g, f are any two selectors of φ, then for any $H \subseteq T$, the function $1_H g + 1_{T \setminus H} f$ is a selector of φ as well. □

Part (a) of Proposition 3 is not new. See Theorem 1 in Podczeck [2008] and Theorem 6.1 in Khan and Sagara [2013].

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Liapounoff's vector measure theorem in Banach spaces

Abstract
We present a result on convexity and weak compactness of the range of a vector measure with values in a Banach space, based on the Maharam classification of measure spaces. Our result extends a recent result of Khan and Sagara [Illinois Journal of Mathematics, forthcoming].

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