University of Innsbruck

Purification and independence

Michael Greinecker, Konrad Podczeck

Working Papers in Economics and Statistics

2013-18

Research Platform eeecon
Empirical and Experimental Economics
University of Innsbruck
http://eeecon.uibk.ac.at/

University of Innsbruck
Working Papers in Economics and Statistics

The series is jointly edited and published by

- Department of Economics
- Department of Public Finance
- Department of Statistics

Contact Address:
University of Innsbruck
Department of Public Finance
Universitaetsstrasse 15
A-6020 Innsbruck
Austria
Tel: + 43 512 507 7171
Fax: + 43 512 507 2970
E-mail: eeecon@uibk.ac.at

The most recent version of all working papers can be downloaded at
http://eeecon.uibk.ac.at/wopec/

For a list of recent papers see the backpages of this paper.

Purification and Independence*

Michael Greinecker[†] and Konrad Podczeck[‡]

July 10, 2013

Abstract

We show that concepts introduced by Aumann more than thirty years ago throw a new light on purification in games with extremely dispersed private information. We show that one can embed payoff-irrelevant randomization devices in the private information of players and use these randomization devices to implement mixed strategies as deterministic functions of the private information. This approach gives rise to very short, elementary, and intuitive proofs for a number of purification results that previously required sophisticated methods from functional analysis or nonstandard analysis. We use our methods to prove a general purification theorem for games with private information in which a player's payoffs can depend in arbitrary ways on events in the private information of other players and in which we allow for shared information in a general way.

1 Introduction

Bayesian decision theory based on expect utility implies that players have no incentives to randomize, because mixed optimal choices must be mixtures over optimal deterministic choices. Applied game theorists have therefore often expressed a preference for equilibria in pure strategies. But the convexifying effect of randomization often guarantees that an equilibrium in mixed strategies exists. Purification theorems allow a researcher to construct pure strategy equilibria out of mixed strategy equilibria. The seminal papers[1] in this category are

*We are grateful to Rabeè Tourky. Discussions with him on existence of pure-strategy equilibria and the relation between randomization and decomposability techniques inspired this research.

[†]University of Innsbruck, michael.greinecker@uibk.ac.at

[‡]University of Vienna, konrad.podczeck@univie.ac.at

[1]We do not discuss purification based on perturbing the game as in Harsanyi [1973] and Govindan et al. [2003]. See Morris [2008] for a comparison of these approaches.

Dvoretzky, Wald, and Wolfowitz [1950], Dvoretzky et al. [1951], Radner and Rosenthal [1982], Milgrom and Weber [1985], and Khan and Rath [2009]. These results are based on players having private information that gives rise to an atomless distribution and action spaces being finite or countably infinite. Mixed strategy equilibria can be shown to exist in great generality,[2] so this approach is quite powerful.

These earlier purification results have never been extended to general compact metric action spaces and, indeed, an example in Khan, Rath, and Sun [1999] shows that this cannot be done when private information is assumed to be merely atomless. The first positive result to overcome this problem was given in Loeb and Sun [2006], where the authors used nonstandard analysis to extend the approach of Dvoretzky-Wald-Wolfowitz to atomless Loeb spaces and general compact metric spaces. They then applied this result to obtain a strong purification result for games with incomplete information in which players have general compact metric action spaces and private information is modeled by a Loeb probability space. It was subsequently shown in Podczeck [2009] that one can replace Loeb spaces by a much larger class of probability spaces he termed super-atomless and showed that theses spaces can be actually characterized as those spaces for which the abstract purification result holds. Mathematically, Podczeck used extreme point methods as introduced in Lindenstrauss [1966]. It was then pointed out in Loeb and Sun [2009] that these general results follow also from the special purification theorem for atomless Loeb spaces by methods developed in Hoover and Keisler [1984]. A further strengthening of these results, based on machinery in Sun [2006] and Podczeck [2010], was given in Wang and Zhang [2012], showing that continuity and compactness assumptions required in earlier results can be dispensed with.

In this paper, we explore an alternative road to purification. In the early days of game theory, mixed strategies have been interpreted as deliberate acts of randomization based on randomization devices. Robert Aumann took the step to explicitly model such randomization devices. In Aumann [1964], he used explicit randomization devices to bypass measurability problems with mixing over measurable functions, and in Aumann [1974], a paper that inspired Radner and Rosenthal [1982], he allowed players to condition on payoff-irrelevant information to study the role of subjectivity and correlation. If this private information includes an atomless field of events independent of the private information of other players, a *secret roulette wheel* in the terminology of Aumann, a player can use these events to implement all mixed strategies as deterministic functions of

[2] See for example Milgrom and Weber [1985] and Al-Najjar and Solan [1999].

the private information. We use these ideas to provide elementary and intuitive proofs of purification when private information is super-atomless.

In Theorem 1 in Section 3, we show, roughly, that one can replace a random probability measure by a random variable whenever the underlying probability space contains a roulette wheel independent of all payoff-relevant information. This is always the case when the underlying probability space is super-atomless. We apply this result to give a short and simple proof of an abstract purification result, Theorem 2, where, as in Wang and Zhang [2012], no compactness or continuity assumptions are involved.

In Section 4, we apply our approach to obtain a purification theorem for games with private information, our Theorem 4. In contrast to all previous purification theorems, a player's payoff may depend in arbitrary ways on the states of the world. We do not require a player's utility function to be measurable with respect to her private information or, in the language of Bayesian games, to depend only on her own type. We also allow for players to receive arbitrary quantitative signals about the private information of other players. We set our purification result in the framework of Aumann [1974], augmented by state dependent utility. This framework incorporates Bayesian games as a special case.[3]

As will become clear, this level of generality is possible by identifying roulette wheels in the private information of players that players can use to mimic mixed strategies. The same idea will illuminate the saturation principle (for probability spaces) in Hoover and Keisler [1984] and we show that it can be seen as a purification principle in its own right.

We want to point out that the classical purification results in Dvoretzky et al. [1950], Dvoretzky et al. [1951], Radner and Rosenthal [1982], Milgrom and Weber [1985], and Khan and Rath [2009] do not directly depend on players being able to condition on payoff-irrelevant information, so our arguments cannot be applied to these settings directly.

[3] A far reaching generalization of Aumann's framework can be found in Grant, Meneghel, and Tourky [2013], where the authors are able to prove the existence of pure strategy equilibria by employing a novel fixed-point theorem from Meneghel and Tourky [2013], in which convexity assumptions are replaced by a decomposability assumption from nonlinear analysis. They obtain pure strategy equilibria in Bayesian games directly without purifying mixed strategy equilibria. It is not clear to us whether a general purification result would hold in their framework, as their assumptions are not directly comparable to ours.

2 Preliminaries

This section contains several mathematical facts we use. These facts are largely well known, we collect them here for ease of reference.

We start with introducing some notation and terminology. If U, V, and W are sets, and $f\colon U \to V$ and $g\colon U \to W$ are functions, then (f, g) denotes the parallel product of f and g; thus $(f, g)(u) = (f(u), g(u))$ for each $u \in U$. By ι_X we denote the identity on a set X.

If Ω is a set and $\mathcal{F} \subseteq 2^{\Omega}$, we denote by $\sigma(\mathcal{F})$ the smallest σ-algebra under set inclusion that contains $\mathcal{F}$ and call it the σ-algebra *generated by* $\mathcal{F}$. A σ-algebra Σ on Ω is *countably generated* if there is a countable family $\mathcal{C}$ such that $\Sigma = \sigma(\mathcal{C})$. If $\langle \Sigma_i \rangle_{i \in I}$ is a family of sub-σ-algebras, we let $\bigvee_{i \in I} \Sigma_i = \sigma\big(\bigcup_{i \in I} \Sigma_i\big)$. If f is a function whose codomain is a measurable space $(X, \mathcal{X})$, the *σ-algebra generated by* f is the σ-algebra $\{f^{-1}(H) : H \in \mathcal{X}\}$ on the domain of f. It is the smallest σ-algebra that makes f measurable.

A probability space is called *super-atomless*[4] if there is no measurable set on which the subspace measure induces a measure algebra which is completely generated by a countable sub-algebra. The canonical example is the fair coin-flipping measure on $\{0, 1\}^{\kappa}$ with κ uncountable. It is a consequence of Maharam's representation theorem that a probability space is super-atomless if and only if there exists an uncountable family of independent random variables on it with uniform distribution on $[0, 1]$ (cf. Lemma 5).

By $\mathcal{B}$ we denote the Borel σ-algebra of $[0, 1]$, and by λ the restriction of Lebesgue measure to $\mathcal{B}$. For a general topological space X, the Borel-σ-algebra of X is denoted by $\mathcal{B}(X)$.

If $(X, \mathcal{X})$ is a measurable space, $\mathcal{M}(X)$ denotes the set of probability measures on $(X, \mathcal{X})$, endowed with the smallest σ-algebra such that for every $B \in \mathcal{X}$ the evaluation function given by $\nu \mapsto \nu(B)$ is measurable.

If X is a topological space, then $\mathcal{M}(X)$ denotes the set of Borel probability measures on X, endowed with the σ-algebra defined in the previous paragraph, substituting $\mathcal{B}(X)$ for $\mathcal{X}$.

A topological space X is called *Souslin* if it is Hausdorff and if there is a continuous surjection from a Polish space onto X. Thus any Polish space is a Souslin space. Recall that if X is a Souslin space, then $\mathcal{B}(X)$ is countably generated. It is not hard to see that this implies that for a Souslin space X, the σ-algebra on $\mathcal{M}(X)$ defined above is countably generated.

[4] Terminology varies. See footnote 4 in Wang and Zhang [2012] for an overview of the various terms that have been used for what we call super-atomless.

We shall make much use of distributions of random measures. Let (Ω, Σ, μ) be a probability space, and $(X, \mathcal{X})$ a measurable space. Let $f : \Omega \to \mathcal{M}(X)$ be measurable. We define the *distribution* μ_f of f by

$$\mu_f(B) = \int_\Omega f(\cdot)(B)\, d\mu$$

for all $B \in \mathcal{X}$.

We shall frequently identify a measurable function with values in X with a measure-valued function whose values are Dirac-measures. It is readily verified that, under this identification, the distribution as defined above coincides with the usual notion of distribution of a measurable function.

Let (Ω, Σ, μ) be a probability space, and $\langle X_i, \mathcal{X}_i \rangle_{i \in I}$ a family of measurable spaces. If $\langle f_i \rangle_{i \in I}$ is a family of measurable functions $f_i : \Omega \to \mathcal{M}(X_i)$, we let the *distribution* or *joint distribution* of $\langle f_i \rangle_{i \in I}$ be the distribution of the function $\otimes_i f_i : \Omega \to \mathcal{M}\big(\prod_{i \in I} X_i\big)$ given by $\otimes_i f_i(\omega) = \otimes_i f_i(\omega)$, where $\prod_{i \in I} X_i$ is endowed with the product-σ-algebra $\bigotimes_{i \in I} \mathcal{X}_i$. A monotone class argument shows that this function is again measurable. If the index i takes on only a small number of values, we shall write something like $(f_1, f_2, \dots, f_n)$ for $\otimes_i f_i$ thus defined. If all f_i are deterministic measurable functions, our notion of joint distribution coincides with the usual one for random variables.

Lemma 1. *Let (Ω, Σ, μ) be a probability space, $(X, \mathcal{X})$ a measurable space, and $\varphi \colon \Omega \times X \to \mathbb{R}$ a $\Sigma \otimes \mathcal{X}$- measurable function. Suppose that for some integrable function $\rho \colon \Omega \to \mathbb{R}_+$, $\sup_{x \in X} |\varphi(\omega, x)| \leqslant \rho(\omega)$ for almost all $\omega \in \Omega$. Then given any measurable function $f \colon \Omega \to \mathcal{M}(X)$,*

$$\int_{\Omega \times X} \varphi(\omega, x) d\mu_{(\iota_\Omega, f)}(\omega, x) = \int_\Omega \int_X \varphi(\omega, x) df(\omega)(x) d\mu(\omega).$$

In particular, if $g \colon \Omega \to X$ is measurable, then

$$\int_{\Omega \times X} \varphi(\omega, x) d\mu_{(\iota_\Omega, g)}(\omega, x) = \int_\Omega \varphi(\omega, g(\omega)) d\mu(\omega).$$

Proof. The statement concerning g is a special case of that concerning f, identifying g with the map $\omega \to \delta_{g(\omega)}$, where $\delta_{g(\omega)}$ denotes Dirac measure at $g(\omega)$. Now as for f, we have

$$\int_{\Omega \times X} |\varphi(\omega, x)| d\mu_{(\iota_\Omega, f)}(\omega, x) \leqslant \int_{\Omega \times X} \rho(\omega) d\mu_{(\iota_\Omega, f)}(\omega, x) = \int_\Omega \rho(\omega) d\mu(\omega) < \infty,$$

so the claim follows from the generalized Fubini theorem for random measures, see Bogachev [2007, Theorem 10.7.2]. □

Lemma 2. *Let* (Ω, Σ, μ) *be a probability space, let* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ *be measurable spaces, and let* $f : \Omega \to X$ *and* $g : \Omega \to Y$ *be independent measurable functions. Then* $\mu_{(f,g)} = \mu_f \otimes \mu_g$.

Proof. It suffices to show that $\mu_{(f,g)}$ and $\mu_f \otimes \mu_g$ agree on all measurable rectangles $A \times B$ in $X \times Y$. Now, $\mu\big((f,g)^{-1}(A \times B)\big) = \mu\big(f^{-1}(A) \cap g^{-1}(B)\big) = \mu\big(f^{-1}(A)\big)\mu\big(g^{-1}(B)\big) = \mu_f(A)\mu_g(B)$. □

The following lemma is fundamental for our constructions. It says that a random probability measure can be seen as the distribution of a random function. According to Rustichini [1993], the result dates back to work of Skorokhod in the 1950s.

Lemma 3. *Let* (Ω, Σ) *be a measurable space, let* X *be a Souslin space, and let* $p : \Omega \to \mathcal{M}(X)$ *be measurable. Then there exists an* $\Sigma \otimes \mathcal{B}$*-measurable mapping* $h : \Omega \times [0, 1] \to X$ *such that for all* $\omega \in \Omega$ *and all* $B \in \mathcal{B}(X)$*,*

$$p(\omega)\big(B\big) = \lambda\Big\{r \in [0, 1] : h(\omega, r) \in B\Big\}$$

Proof. Bogachev [2007, Proposition 10.7.6], or Aumann [1964, Lemma F]. □

Lemma 4. *Let* $(X, \mathcal{X})$ *and* $(Y, \mathcal{Y})$ *be measurable spaces and let* $\mathcal{A}$ *be a countably generated sub-σ-algebra of* $\mathcal{X} \otimes \mathcal{Y}$*. Then there exists a countably generated sub-σ-algebra* $\mathcal{C}$ *of* $\mathcal{X}$ *such that* $\mathcal{A} \subseteq \mathcal{C} \otimes \mathcal{Y}$.

Proof. We use the easily proved and well known fact that if a set is in the σ-algebra generated by some family, it is already in the σ-algebra generated by a countable sub-family. Thus let $\mathcal{G}$ be a countable family generating for $\mathcal{A}$. By the fact just stated, for each $G \in \mathcal{G}$ we can find a countable family $\mathcal{R}_G$ of measurable rectangles in $\mathcal{X} \otimes \mathcal{Y}$ such that G is in the σ-algebra generated by $\mathcal{R}_G$. Let π_X be the projection of $X \times Y$ onto X. Then $\bigcup_{G \in \mathcal{G}} \pi_X(\mathcal{R}_G)$ is countable, and we can take $\mathcal{C}$ to be the σ-algebra generated by this family. □

Lemma 5. *Let* (Ω, Σ, μ) *be a probability space. Then:*

(a) μ *is atomless if and only if there is an infinite independent family* $\langle E_i \rangle_{i \in I}$ *in* Σ *with* $\mu(E_i) = 1/2$ *for each* $i \in I$.

(b) μ *is super-atomless if and only if there is an uncountable independent family* $\langle E_i \rangle_{i \in I}$ *in* Σ *with* $\mu(E_i) = 1/2$ *for each* $i \in I$.

Proof. (a) follows from the fact that a probability space (Ω, Σ, μ) is atomless if and only if there is a map $h \colon \Omega \to \{0,1\}^{\omega}$ whose distribution is the fair coin-flipping measure. For (b), see Podczeck [2010, Remark 1 and Lemma 2]. $\square$

Lemma 6. *Let (Ω, Σ, μ) be a probability space, and Σ', Σ_1 two sub-σ-algebra of Σ. Suppose Σ_1 is countably generated and that $\mu \restriction \Sigma'$ is super-atomless. Then there is a countably generated sub-σ-algebra $\Sigma_2 \subseteq \Sigma'$ which is independent of Σ_1 and such that $\mu \restriction \Sigma_2$ is atomless.*

Proof. By Lemma 5(b) there is an uncountable independent family $\langle E_i \rangle_{i \in I}$ in Σ' with $\mu(E_i) = 1/2$ for each $i \in I$. Now by Fremlin [2010, Theorem 272Q], there is a countable $H \subseteq I$ such that the σ-algebras Σ_1 and $\sigma(\{E_i \colon i \in I \backslash H\})$ are independent. As I is uncountable, there is a countable infinite set $J \subseteq I \backslash H$. Let $\Sigma_2 = \sigma(\{E_i \colon i \in J\})$. In view of Lemma 5(a), Σ_2 is as required. $\square$

3 Purification in an abstract setting

Purification theorems in the tradition of Dvoretzky-Wald-Wolfowitz show that a family of functions evaluated with respect to a deterministic function agrees with the evaluation with respect to a given random measure. By Lemma 1, this can be reduced to showing that the induced measures on a product space agree on a σ-algebra that makes all these functions measurable. If there is a σ-algebra which is independent of the former and on which μ is atomless, we can use it as a roulette wheel. It is a randomization device that can be used to make the implicit randomization in a random measure explicit by using Lemma 3.

Theorem 1. *Let (Ω, Σ, μ) be a probability space and Σ_1 and Σ_2 be independent sub-σ-algebras of Σ such that μ is atomless on Σ_2. Let X be a Souslin space, and $f : \Omega \to \mathcal{M}(X)$ a Σ_1-measurable function. Then there exists a measurable function $g : \Omega \to X$ such that the restriction of $\mu_{(\iota_\Omega, g)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$ coincides with the restriction of $\mu_{(\iota_\Omega, f)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$.*

Proof. Choose a Σ_2-measurable map $w : \Omega \to [0,1]$ with $\mu_w = \lambda$, as is possible because μ is atomless on Σ_2, and let $h : \Omega \times [0,1] \to X$ be a $\Sigma_1 \otimes \mathcal{B}$-measurable map chosen according to Lemma 3. Now let $g = h \circ (\iota_\Omega, w)$. To see that (ι_Ω, g) has the desired distribution, it suffices to consider a rectangle $A \times B \in \Sigma_1 \times \mathcal{B}(X)$.

By Lemma 2, $\mu_{(\iota_\Omega, w)} \restriction \Sigma_1 \otimes \mathcal{B} = \mu \restriction \Sigma_1 \otimes \lambda$. Thus, using Fubini's theorem,

$$\begin{aligned}
\int_A f(\omega)(B) d\mu(\omega) &= \int_A \lambda\big(h(\omega, \cdot)^{-1}(B)\big) d\mu(\omega) \\
&= \mu \restriction \Sigma_1 \otimes \lambda\big((A \times [0,1]) \cap h^{-1}(B)\big) \\
&= \mu_{(\iota_\Omega, w)} \restriction \Sigma_1 \otimes \mathcal{B}\big((A \times [0,1]) \cap h^{-1}(B)\big) \\
&= \mu\big((\iota_\Omega, w)^{-1}\big((A \times [0,1]) \cap h^{-1}(B)\big) \\
&= \mu\big((\iota_\Omega, w)^{-1}(A \times [0,1]) \cap (\iota_\Omega, w)^{-1}(h^{-1}(B))\big) \\
&= \mu(A \cap g^{-1}(B)) \\
&= \mu_{(\iota_\Omega, g)}(A \times B). \qquad \square
\end{aligned}$$

As a corollary, we get a generalization of the main theorem of Podczeck [2009] and Loeb and Sun [2009]:

Theorem 2. *Let* (Ω, Σ, μ) *be a probability space, let* X *be a Souslin space, and let* $f : \Omega \to \mathcal{M}(X)$ *be a measurable function. Let* J *be a countable set, and for each* $j \in J$ *let* $\varphi_j \colon \Omega \times X \to \mathbb{R}$ *be a jointly measurable mapping such that for some integrable function* $\rho \colon \Omega \to \mathbb{R}_+$, $\sup_{x \in X} |\varphi_j(\omega, x)| \leqslant \rho_j(\omega)$ *for almost all* $\omega \in \Omega$. *Suppose* μ *is super-atomless. Then there is a function* $g \colon \Omega \to X$ *such that*

(a) g *is measurable;*

(b) $\int_\Omega \int_X \varphi_j(\omega, x) df(\omega)(x) d\mu(\omega) = \int_\Omega \varphi_j(\omega, g(\omega)) d\mu(\omega)$ *for all* $j \in J$.

Proof. The σ-algebra on $\Omega \times X$ generated by the countable family $\langle \varphi_j \rangle_{j \in J}$ of real-valued functions is countably generated. It follows from Lemma 4 that Σ has a countably generated sub-σ-algebra Σ_1 such that all these functions are $\Sigma_1 \otimes \mathcal{B}(X)$-measurable. By Lemma 6, there exists a sub-σ-algebra Σ_2 that is independent of Σ_1 and on which μ is atomless. The claim now follows from Theorem 1 in conjunction with Lemma 1. $\square$

As was pointed out in Wang and Zhang [2012], in a context like that of Theorem 2 there must actually exist uncountably many purifications. This may also be seen from our proof: If $\langle \Sigma_i \rangle_{i \in I}$ is any countable and independent family of sub-σ-algebras of Σ such that each Σ_i is countably generated and independent of Σ_1, and such that μ is atomless on each Σ_i, then Lemma 6 applied to $\Sigma_1 \vee \bigvee_{i \in I} \Sigma_i$ gives another countably generated sub-σ-algebra of Σ, say Σ_+, such that Σ_+ is independent of Σ_1 and each Σ_i, and such that $\mu \restriction \Sigma_+$ is again atomless. Now such a Σ_+ leads to another function w as used to construct g in the proof of Theorem 1, and leads thus to another purification in the context of Theorem 2.

It is sometimes useful to have purifications that satisfy additional constraints. For example, actions available to a player in a Bayesian game might be type-dependent and then we want only purified strategies that are actually feasible. The following theorem establishes this.[5] In this theorem, G_Γ denotes the graph of the correspondence Γ.

Theorem 3. *Let (Ω, Σ, μ) be a probability space, X a Souslin space, and let $\Gamma\colon \Omega \to 2^X$ be a correspondence which admits a measurable selection and such that $G_\Gamma \in \Sigma \otimes \mathcal{B}(X)$. Let $f : \Omega \to \mathcal{M}(X)$ be a measurable function such that $\mu_{(\iota_\Omega, f)}(G_\Gamma) = 1$. Endow G_Γ with the subspace-σ-algebra defined from $\Sigma \otimes \mathcal{B}(X)$. Let J be a countable set, and for each $j \in J$ let $\varphi_j\colon \Gamma \to \mathbb{R}$ be a measurable map such that for some integrable $\rho\colon \Omega \to \mathbb{R}_+$, $\sup_{x \in \Gamma(\omega)} |\varphi_j(\omega, x)| \leqslant \rho_j(\omega)$ for almost all $\omega \in \Omega$. Suppose μ is super-atomless. Then there is a $g\colon \Omega \to X$ such that*

(a) g *is a measurable selection of* Γ;

(b) $\int_\Omega \int_X \varphi_j(\omega, x) df(\omega)(x) d\mu(\omega) = \int_\Omega \varphi_j(\omega, g(\omega)) d\mu(\omega)$ *for all* $j \in J$.

Remark 1. Sufficient conditions for the existence of a measurable selection as required in this theorem are for example that (Ω, Σ, μ) is complete and Γ is nonempty-valued [Castaing and Valadier, 1977, Theorem III.22], or that X is Polish and Γ a measurable correspondence with closed and nonempty values [Castaing and Valadier, 1977, Theorem III.8].

Proof of Theorem 3. For all $j \in J$, extend ϕ_j to all of $\Omega \times X$, setting $\phi_j(\omega, x) = 0$ for $(\omega, x) \notin \Gamma$. We can now use Theorem 2 to obtain a measurable function $\tilde{g} : \Omega \to X$ such that (b) and in addition

$$\int_{\Omega \times X} 1_{G_\Gamma} d\mu_{(\iota_\Omega, \tilde{g})} = \int_{\Omega \times X} 1_{G_\Gamma} d\mu_{(\iota_\Omega, f)}$$

holds. By hypothesis, $\int 1_{G_\Gamma} d\mu_{(\iota_\Omega, f)} = \mu_{(\iota_\Omega, f)}(\Gamma) = 1$, and it follows that there is $N \in \Sigma$ with $\mu(N) = 0$ such that $\tilde{g}(\omega) \in \Gamma(\omega)$ for all $\omega \in \Omega \setminus N$. Let $s : \Omega \to X$ be a measurable selection of Γ. Define $g : \Omega \to X$ by

$$g(\omega) = \begin{cases} \tilde{g}(\omega) & \text{if } \omega \in \Omega \setminus N \\ s(\omega) & \text{if } \omega \in N. \end{cases}$$

Then g is a measurable selection of Γ. Since $g = \tilde{g}$ almost surely, the integrals in (b) will be unchanged if we replace $\tilde{g}$ by g. □

[5] This result generalizes Theorem 15 in Carmona and Podczeck [2013], which is used there for purifying mixed equilibria in games with a continuum of players.

4 Purification in games with private information

In this section, we prove a general purification theorem for games with private information. Our model is a slight modification of the extremely elegant framework in Aumann [1974]. The main difference is that we allow for state dependent preferences and assume that all players have the same prior. There is a probability space (Ω, Σ, μ) of *states of the world* with μ being the common prior of all players. There is a finite set N of *players*. For each player $i \in N$, there is an *action space* A which is a Souslin *action space*. We let $A = \prod_{i \in N} A_i$. Also, for each player $i \in N$ there is a bounded and measurable *utility function* $u_i : \Omega \times A \to \mathbb{R}$.[6] The information of each player is given by a sub-σ-algebra $\mathcal{I}_i \subseteq \Sigma$. The idea is that nature picks a state according to μ and then every player is informed about the events in her σ-algebra that contain the state. Players receive private information about the state and then choose their actions.

A *mixed strategy* of $i \in N$ is a $\mathcal{I}_i$-measurable function $m_i : \Omega \to \mathcal{M}(A_i)$. A *pure strategy* of $i \in N$ is a $\mathcal{I}_i$-measurable function $p_i : \Omega \to A_i$. Clearly, we can treat pure strategies as degenerate mixed strategies. We assume the following:

(A) There are a families $\langle \mathcal{P}_i \rangle_{i \in N}$ and $\langle \mathcal{S}_i \rangle_{i \in N}$ of sub-σ-algebras of Σ such that:

(i) $\mathcal{I}_i = \mathcal{P}_i \vee \mathcal{S}_i$ for each $i \in N$.

(ii) For each $i \in N$, $\mathcal{S}_i$ is independent of $\bigvee_{j \neq i} \mathcal{S}_j$.

(iii) $\mu \restriction \mathcal{S}_i$ is super-atomless for each $i \in N$.

(iv) $\mathcal{P}_i$ is countably generated for each $i \in N$.

We think of $\mathcal{S}_i$ as the secret private information of player i and $\mathcal{P}_i$ as the part of her information other players may be informed about. But $\mathcal{S}_i$ may not be completely secret. It is not possible to infer anything about $\mathcal{S}_i$ from the pooled secret information of other players $\bigvee_{j \neq i} \mathcal{S}_i$, but we allow for any dependence between $\mathcal{S}_i$ and the $\mathcal{P}_j$. In particular, every countable family of numerical signals a player receives about the information of another player is admissible. In Loeb and Sun [2006] and Khan and Zhang [2012] it is required that all private information is mutually independent conditional on a countably valued random variable. We made our independence assumption on the $\mathcal{S}_i$ in unconditional form because there is no sensible notion of a random variable we should condition on.

[6] We actually never use boundedness, but it ensures that expected utility is well defined. Clearly, weaker assumptions would do.

We make no assumptions on the utility functions, they can depend in arbitrary ways on both actions and states. In contrast, in Loeb and Sun [2006] and Khan and Zhang [2012] it is required that payoffs depend only on private information and some countably valued random variable.

The reason why we can work with very weak independence assumptions is the following Lemma. Intuitively, it says that if μ is super-atomless on a σ-algebra independent of another σ-algebra, then a large number of events in the former σ-algebra remain independent of the latter after conditioning on an arbitrary countably generated σ-algebra.

Lemma 7. *Let (Ω, Σ, μ) be a probability space, and $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{P}$ sub-σ-algebras of Σ. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are independent, that $\mu\restriction\mathcal{F}_2$ is super-atomless, and that $\mathcal{P}$ is countably generated. Then there is a sub-σ-algebra $\mathcal{F}_3 \subseteq \mathcal{F}_2$ such that $\mathcal{F}_3$ and $\mathcal{F}_1 \vee \mathcal{P}$ are independent and such that $\mu\restriction\mathcal{F}_3$ is super-atomless*

Proof. By Lemma 5 there is an uncountable independent family $\langle E_i\rangle_{i\in I}$ in $\mathcal{F}_2$ with $\mu(E_i) = 1/2$ for each $i \in I$. Let $\mathcal{F} = \sigma\big(\langle E_i\rangle_{i\in I}, \mathcal{F}_1\big)$. Let $\mathcal{C} \subseteq \mathcal{P}$ be a countable algebra generating $\mathcal{P}$, and for each $C \in \mathcal{C}$ let h_C be a conditional expectation of C on $\mathcal{F}$. Let $\mathcal{H} \subseteq \mathcal{F}$ be a countably generated sub-σ-algebra such that each h_C is $\mathcal{H}$-measurable.

There is a countable $J \subseteq I$ such that $\mathcal{H} \subseteq \sigma\big(\langle E_i\rangle_{i\in I\setminus J}, \mathcal{F}_1\big)$. Set $H = I\setminus J$, let $\mathcal{F}_3 = \sigma\big(\langle E_i\rangle_{i\in H}\big)$, and let $\mathcal{F}_4 = \sigma\big(\langle E_i\rangle_{i\in J}, \mathcal{F}_1\big)$. Observe that $\mathcal{F}_3$ and $\mathcal{F}_4$ are independent [use Fremlin, 2010, 272F and 272K]. In particular, therefore, $\int_G h = \mu(G)\int_\Omega h$ whenever $G \in \mathcal{F}_3$ and h is $\mathcal{F}_4$-measurable.

Pick any $G \in \mathcal{F}_3$, $F \in \mathcal{F}_1$ and $C \in \mathcal{C}$. By the last sentence of the previous paragraph, as both $\mathcal{H} \subseteq \mathcal{F}_4$ and $\mathcal{F}_1 \subseteq \mathcal{F}_4$, we have

$$\int_{G\cap F} h_C = \int_G 1_F h_C = \mu(G)\int_\Omega 1_F h_C = \mu(G)\mu(F\cap C)$$

where the last equality holds because $F \in \mathcal{F}$ and h_C is a conditional expectation of C on $\mathcal{F}$. On the other hand, by this property of h_C, as also $G\cap F \in \mathcal{F}$,

$$\int_{G\cap F} h_C = \mu(G\cap F\cap C).$$

It follows that each $G \in \mathcal{F}_3$ is independent of each intersection of finitely many elements of $\mathcal{F}_1 \cup \mathcal{C}$. Now by a monotone class argument, it follows that each $G \in \mathcal{F}_3$ is independent of every element of $\sigma\big(\mathcal{F}_1 \cup \mathcal{C}\big)$, i.e., $\mathcal{F}_3$ and $\sigma\big(\mathcal{F}_1 \cup \mathcal{C}\big)$ are independent. Finally, just note that $\sigma\big(\mathcal{F}_1 \cup \mathcal{C}\big) = \mathcal{F}_1 \vee \mathcal{P}$. □

Before we can state and prove our main theorem, we introduce another Lemma that gives us a variant of Theorem 1. The proof can be found in an appendix.

Lemma 8. *Let (Ω, Σ, μ) be a probability space, let Σ_1, Σ_2, $\mathcal{I}$ be sub-σ-algebra of Σ, let X, Y be Souslin spaces, and $f_x \colon \Omega \to \mathcal{M}(X)$ a $\mathcal{I}$-measurable function. Suppose that Σ_1 and Σ_2 are independent, that $\Sigma_2 \subseteq \mathcal{I}$, and that $\mu \restriction \Sigma_2$ is atomless. Then there is a $\mathcal{I}$-measurable function $g_x \colon \Omega \to X$ such that whenever $f_y \colon \Omega \to \mathcal{M}(Y)$ is Σ_1-measurable, then $\mu_{(\iota_\Omega, f_y, g_x)}$ and $\mu_{(\iota_\Omega, f_y, f_x)}$ agree on $\Sigma_1 \otimes \mathcal{B}(Y) \otimes \mathcal{B}(X)$.*

Our purification theorem shows that a player's mixed strategy can be replaced by a pure strategy that depends in exactly the same way on all events that determine payoffs, actions distributions, and shared information. Strategically, they are essentially indistinguishable:

Theorem 4. *(a) There exists a countably generated sub-σ-algebra Σ' of Σ such that u_i is $\Sigma' \otimes \mathcal{B}(A)$-measurable for all $i \in N$. (b) For any player $i \in N$ and any mixed strategy m_i of i, there is a pure strategy p_i such that given any profile $\langle m_j \rangle_{j \neq i}$ of mixed strategies of the other players the joint distributions of $\big(\iota_\Omega, \langle m_j \rangle_{j \in N}\big)$ and $\big(\iota_\Omega, \langle p_i, m_j \rangle_{j \neq i}\big)$ agree on $\Sigma' \otimes \mathcal{B}(A)$.*

Proof. By Lemma 4, for each player $i \in N$ there is a countably generated σ-algebra $\Sigma^i \subseteq \Sigma$ such that u_i is $\Sigma^j \otimes \mathcal{B}(A)$-measurable. Set $\Sigma' = \bigvee_{i \in N} \Sigma^i$, so that Σ' is as required in (a).

Fix any player i and any mixed strategy m_i of i. Let

$$\Sigma_1 = \Sigma' \vee \bigvee_{j \neq i} \mathcal{P}_j \vee \bigvee_{j \neq i} \mathcal{S}_j.$$

By Lemma 7 and condition (A), there is a σ-algebra $\Sigma_2 \subseteq \mathcal{I}_i$ which is independent of Σ_1 and such that $\mu \restriction \Sigma_2$ is atomless. Note that any strategy m_j of any player $j \neq i$ must be Σ_1-measurable. The claim now follows from Lemma 8, substituting $\mathcal{I}_i$ for $\mathcal{I}$, m_i for f_x, and $\otimes_{j \neq i} m_j$ for f_y, noting that $\mathcal{B}(A) = \prod_{j \in I} \mathcal{B}(A_j)$, as all the spaces A_j are Souslin. □

Theorem 4 guarantees that not only the distributions of action profiles and the distributions over realized payoffs coincide for the purified and the original strategies, but also the joint distributions of action profiles and realized payoffs of these two strategies. Our notion of purification is therefore stronger than the notion of strong purification introduced in Khan et al. [2006], which is already strong enough to construct pure strategy Nash equilibria from mixed equilibria.

The remarkable part of our proof is that it depends on player i being able to condition on events in a σ-algebra that is completely payoff-irrelevant, independent of the private information of everyone else, and rich enough that μ is atomless on it. In the terminology of Aumann [1974], such a σ-algebra is a secret roulette wheel. Conceptually, there is no reason to draw a distinction between a mixed strategy and a deterministic function of a secretly used chance device. It is worth pointing out that this point does not hold for papers in which private information is merely assumed to be atomless and action spaces are countable. In that case, u_i may well generate $\mathcal{I}_i \otimes \mathcal{B}(A)$.

5 Saturation revisited

The following saturation principle was introduced in Hoover and Keisler [1984] and used as a basis for proving a purification result in Loeb and Sun [2009]: A probability space (Ω, Σ, μ) is *saturated* if for every two Polish spaces X and Y, every probability measure $\nu \in \mathcal{M}(X \times Y)$ with marginal ν_X, and every random variable $\phi : \Omega \to X$, there is a random variable $\psi : \Omega \to Y$ such that (ϕ, ψ) has distribution ν. It was shown implicitly in Hoover and Keisler [1984] and more explicitly in Fajardo and Keisler [2002, Theorem B.7] that a probability space is saturated if and only if it is super-atomless.[7] Actually, saturation can be seen as a form of automatic purification. We first relate the saturation principle, strengthened to Souslin spaces, to independent randomization by employing Theorem 1, in order to show what happens in the background when saturation is employed as a black box.

Proposition 1. *Let (Ω, Σ, μ) be a probability space, X and Y be Souslin spaces, $\nu \in \mathcal{M}(X \times Y)$ and $\phi : \Omega \to X$ be a random variable with distribution equal to the X-marginal ν_X of ν. Suppose there exists a sub-σ-algebra Σ_2 of Σ on which μ is atomless and such that Σ_2 is independent of the σ-algebra generated by ϕ. Then there exists a random variable $\psi : \Omega \to Y$ such that (ϕ, ψ) has distribution ν.*

Proof. Since regular conditional probabilities are guaranteed to exist in Souslin spaces [Bogachev, 2007, Corollary 10.4.6] and the product of Souslin spaces is again a Souslin space [Bogachev, 2007, Lemma 6.6.5(iii)] there exists a measurable function $r : X \to \mathcal{M}(Y)$ such that (ι_X, r) has distribution ν when we endow X with the marginal measure ν_X. Let Σ_1 be the σ-algebra generated by ϕ. Let $f = r \circ \phi$. We can now take ψ to be the g guaranteed to exist by Theorem 1. Then the distribution of (ι_Ω, ψ) restricted to $\Sigma_1 \otimes \mathcal{B}(Y)$ coincides with the distribution of $(\iota_\Omega, r \circ \phi)$ restricted to $\Sigma_1 \otimes \mathcal{B}(Y)$. Since ϕ is Σ_1-measurable, the distribution of $(\iota_\Omega, \phi, \psi)$ restricted to $\Sigma_1 \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$ coincides with the distribution of $(\iota_\Omega, \phi, r \circ \phi)$ restricted to $\Sigma_1 \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$. For let $A \times B \times C$ be a rectangle in $\Sigma_1 \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$. We have $\mu_{(\iota_\Omega, \phi, \psi)}(A \times B \times C) = \mu_{(\iota_\Omega, \psi)}\big(A \cap \phi^{-1}(B) \times C\big)$ $= \mu_{(\iota_\Omega, r\circ\phi)}\big(A \cap \phi^{-1}(B) \times C\big) = \mu_{(\iota_\Omega, \phi, r\circ\phi)}(A \times B \times C)$. Since $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ equals $\mathcal{B}(X \times Y)$, (ϕ, ψ) has distribution ν. □

[7] The notion of saturation for adapted processes introduced in Hoover and Keisler [1984] gives rise to a much more restricted class of probability spaces.

The following proposition shows that joint applications of Theorem 1 and Lemma 6 could be based on saturation directly. We therefore think of saturation as a form of automatic purification.

Proposition 2. *Let (Ω, Σ, μ) be a saturated probability space, Σ_1 a countably generated sub-σ-algebra, X a Souslin space and $f : \Omega \to \mathcal{M}(X)$ be Σ_1- measurable. Then there exists a function $g : \Omega \to X$ such that the restriction of $\mu_{(\iota_\Omega, g)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$ coincides with the restriction of $\mu_{(\iota_\Omega, f)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$.*

Proof. Since Σ_1 is countably generated, there exists a measurable function $\phi : \Omega \to \{0, 1\}^\omega$ such that Σ_1 is the σ-algebra generated by ϕ. Let ν be the distribution of (ϕ, f).

By saturation, there exists a measurable function $g : \Omega \to X$ such that (ϕ, g) has distribution ν. But this implies that the restriction of $\mu_{(\iota_\Omega, g)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$ coincides with the restriction of $\mu_{(\iota_\Omega, f)}$ to $\Sigma_1 \otimes \mathcal{B}(X)$. Indeed, let $A \in \Sigma_1$ and $B \in \mathcal{B}(X)$. Since ϕ generates Σ_1, there is a Borel set $C \in \mathcal{B}\big(\{0, 1\}^\omega\big)$ such that $A = \phi^{-1}(C)$. It follows that $\mu_{(\iota_\Omega, g)}(A \times B) = \mu(\phi^{-1}(C) \cap g^{-1}(B)) = \nu(C \times B)$. Since $\nu(C \times B) = \mu_{(\phi, f)}(C \times B) = \mu_{(\iota_\Omega, f)}\big(\phi^{-1}(C) \times B\big) = \mu_{(\iota_\Omega, f)}(A \times B)$, the result follows. $\square$

6 Conclusion

The intuition behind our results is very simple: A super-atomless probability space contains a very large number of nontrivial independent events, but only a small number of events are important in determining the payoffs being realized and actions being played. So a lot of events serve no relevant function and can be used as randomization devices. This rich supply of independent events makes purification with super-atomless spaces qualitatively different from purification with arbitrary atomless spaces of private information and finite or countably infinite action spaces.

It is the rich supply of independent events in a super-atomless probability space that allows for a lot of constructions not available for general atomless probability spaces. One of the most striking consequences of this richness for economic applications is the existence of nontrivial jointly measurable processes that allow for enough independence to guarantee an exact law of large numbers as provided by Sun [2006] to hold. Such processes were shown to exist in Sun [2006] by nonstandard methods. General existence results were given in Podczeck [2010], where explicit use of the rich independence structure of super-atomless probability spaces is made.

From a game theoretic point of view, the independence structure of super-atomless probability spaces may well be too rich. Our Lemma 8 showed that most events in an independent σ-algebra on which a probability measure is atomless, remain completely hidden given any natural notion of quantitative tangible signals. We suggest that future research concentrates on finding pure equilibria in games with simpler private information by exploiting concrete structures. The purifications obtained from super-atomless probability spaces are not pure enough, they are mixed strategies in the sense of Aumann.

Appendix

Proof of Lemma 8. As in the proof of Theorem 1, choose a Σ_2-measurable map $w\colon \Omega \to [0,1]$ so that $\mu_w = \lambda$, and by Lemma 3, choose a $\mathcal{I} \otimes \mathcal{B}$ measurable $h\colon \Omega \times [0,1] \to X$ so that $f_x(B_x) = \lambda(h(\omega,\cdot)^{-1}(B_x))$ for each $B_x \in \mathcal{B}(X)$. Let $g_x = h \circ (\iota_\Omega, w)$ and note that g_x is $\mathcal{I}$-measurable, as $\Sigma_2 \subseteq \mathcal{I}$.

Now pick any Σ_1-measurable function $f_y : \Omega \to \mathcal{M}(Y)$. Note that by Lemma 2, $\mu_{(\iota_\Omega, f_y, w)} \restriction \Sigma_1 \otimes \mathcal{B}(Y) \otimes \mathcal{B} = \big(\mu_{(\iota_\Omega, f_y)} \restriction \Sigma_1 \otimes \mathcal{B}(Y)\big) \otimes \lambda$.

Now given any rectangle $A \times B_y \times B_x \in \Sigma_1 \otimes \mathcal{B}(Y) \otimes \mathcal{B}(X)$, we can calculate as follows, where the sixth equality follows by Fubini's theorem, the seventh by the generalized version of Fubini's theorem, and where $\tilde{h}\colon \Omega \times Y \times [0,1] \to X$ is given by setting $\tilde{h}(\omega, y, r) = h(\omega, r)$, and δ is used to denote a Dirac measure:

$$\begin{aligned}
\mu_{(\iota_\Omega, f_y, g_x)}(A \times B_y \times B_x) &= \int_A f_y(\omega)(B_y)\delta_{g_x(\omega)}(B_x)d\mu(\omega) \\
&= \int_A f_y(\omega)(B_y)\delta_{h(\omega, w(\omega))}(B_x)d\mu(\omega) \\
&= \int_A f_y(\omega)(B_y)\delta_{(\iota_\Omega(\omega), w(\omega))}(h^{-1}(B_x))d\mu(\omega) \\
&= \mu_{(\iota_\Omega, f_y, w)} \restriction \Sigma_1 \otimes \mathcal{B}(Y) \otimes \mathcal{B}\big(\tilde{h}^{-1}(B_x) \cap (A \times B_y \times [0,1])\big) \\
&= \big(\mu_{(\iota_\Omega, f_y)} \restriction \Sigma_1 \otimes \mathcal{B}(Y)\big) \otimes \lambda\big(\tilde{h}^{-1}(B_x) \cap (A \times B_y \times [0,1])\big) \\
&= \int_{A \times B_y} \lambda(h(\omega,\cdot)^{-1}(B_x))d\mu_{(\iota_\Omega, f_y)} \\
&= \int_A f_y(\omega)(B_y)\lambda(h(\omega,\cdot)^{-1}(B_x))d\mu(\omega) \\
&= \int_A f_y(\omega)(B_y)f_x(\omega)(B_x)d\mu(\omega) \\
&= \mu_{(\iota_\Omega, f_y, f_x)}(A \times B_y \times B_x).
\end{aligned}$$

□

References

Nabil I Al-Najjar and Eilon Solan. *Equilibrium Existence in Games with Incomplete Information: The Countable Case*. Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1999.

Robert J. Aumann. Mixed and behavior strategies in infinite extensive games. In *Advances in Game Theory*, pages 627–650. Princeton Univ. Press, Princeton, N.J., 1964.

Robert J. Aumann. Subjectivity and correlation in randomized strategies. *J. Math. Econom.*, 1(1):67–96, 1974. ISSN 0304-4068.

Vladimir I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007. ISBN 978-3-540-34513-8; 3-540-34513-2. doi: 10.1007/978-3-540-34514-5. URL http://dx.doi.org/10.1007/978-3-540-34514-5.

Guilherme Carmona and Konrad Podczeck. Existence of nash equilibrium in games with a measure space of players and discontinuous payoff functions. mimeo, 2013.

Charles Castaing and Michel Valadier. *Convex analysis and measurable multifunctions*. Lecture Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin, 1977.

Aryeh Dvoretzky, Abraham Wald, and Jacob Wolfowitz. Elimination of randomization in certain problems of statistics and of the theory of games. *Proc. Nat. Acad. Sci. U. S. A.*, 36:256–260, 1950. ISSN 0027-8424.

Aryeh Dvoretzky, Abraham Wald, and Jacob Wolfowitz. Elimination of randomization in certain statistical decision procedures and zero-sum two-person games. *Ann. Math. Statistics*, 22:1–21, 1951. ISSN 0003-4851.

Sergio Fajardo and H. Jerome Keisler. *Model theory of stochastic processes*, volume 14 of *Lecture Notes in Logic*. Association for Symbolic Logic, Urbana, IL, 2002. ISBN 1-56881-167-5; 1-56881-172-1.

David H. Fremlin. *Measure theory. Vol. 2: Broad foundations*. Torres Fremlin, Colchester, 2010. ISBN 978-0-9538129-7-4. Broad foundations, second Editionl.

Srihari Govindan, Philip J. Reny, and Arthur J. Robson. A short proof of Harsanyi's purification theorem. *Games Econom. Behav.*, 45(2):369–374, 2003. ISSN 0899-8256. doi: 10.1016/S0899-8256(03)00149-0. URL http://dx.doi.org/10.1016/S0899-8256(03)00149-0. Special issue in honor of Robert W. Rosenthal.

Simon Grant, Idione Meneghel, and Rabee Tourky. Savage games. http://ssrn.com/abstract=2281607, 2013.

John C. Harsanyi. Games with randomly disturbed payoffs: a new rationale for mixed-strategy equilibrium points. *Internat. J. Game Theory*, 2(1):1–23, 1973. ISSN 0020-7276.

Douglas N. Hoover and H. Jerome Keisler. Adapted probability distributions. *Trans. Amer. Math. Soc.*, 286(1):159–201, 1984. ISSN 0002-9947. doi: 10.2307/1999401. URL http://dx.doi.org/10.2307/1999401.

M. Ali Khan and Kali P. Rath. On games with incomplete information and the Dvoretsky-Wald-Wolfowitz theorem with countable partitions. *J. Math. Econom.*, 45(12):830–837, 2009. ISSN 0304-4068. doi: 10.1016/j.jmateco.2009.06.003. URL http://dx.doi.org/10.1016/j.jmateco.2009.06.003.

M Ali Khan and Yongchao Zhang. On sufficiently diffused information and finite-player games with private information. Technical report, working paper, Johns Hopkins University, 2012.

M. Ali Khan, Kali P. Rath, and Yeneng Sun. On a private information game without pure strategy equilibria. *J. Math. Econom.*, 31(3):341–359, 1999. ISSN 0304-4068. doi: 10.1016/S0304-4068(97)00063-3. URL http://dx.doi.org/10.1016/S0304-4068(97)00063-3.

M. Ali Khan, Kali P. Rath, and Yeneng Sun. The Dvoretzky-Wald-Wolfowitz theorem and purification in atomless finite-action games. *Internat. J. Game Theory*, 34(1):91–104, 2006. ISSN 0020-7276. doi: 10.1007/s00182-005-0004-3. URL http://dx.doi.org/10.1007/s00182-005-0004-3.

Joram Lindenstrauss. A short proof of Liapounoff's convexity theorem. *J. Math. Mech.*, 15:971–972, 1966.

Peter Loeb and Yeneng Sun. Purification of measure-valued maps. *Illinois J. Math.*, 50(1-4):747–762 (electronic), 2006. ISSN 0019-2082. URL http://projecteuclid.org/getRecord?id=euclid.ijm/1258059490.

Peter Loeb and Yeneng Sun. Purification and saturation. *Proc. Amer. Math. Soc.*, 137(8):2719–2724, 2009. ISSN 0002-9939. doi: 10.1090/S0002-9939-09-09818-9. URL http://dx.doi.org/10.1090/S0002-9939-09-09818-9.

Idione Meneghel and Rabee Tourky. A fixed point theorem for closed-graphed decomposable-valued correspondences. *ArXiv e-prints*, June 2013.

Paul R. Milgrom and Robert J. Weber. Distributional strategies for games with incomplete information. *Math. Oper. Res.*, 10(4):619–632, 1985. ISSN 0364-765X. doi: 10.1287/moor.10.4.619. URL http://dx.doi.org/10.1287/moor.10.4.619.

Stephen Morris. Purification. In Steven N. Durlauf and Lawrence E. Blume, editors, *The New Palgrave Dictionary of Economics*. Palgrave Macmillan, Basingstoke, 2008.

Konrad Podczeck. On purification of measure-valued maps. *Econom. Theory*, 38(2):399–418, 2009. ISSN 0938-2259. doi: 10.1007/s00199-007-0319-3. URL http://dx.doi.org/10.1007/s00199-007-0319-3.

Konrad Podczeck. On existence of rich Fubini extensions. *Econom. Theory*, 45 (1-2):1–22, 2010. ISSN 0938-2259. doi: 10.1007/s00199-009-0458-9. URL http://dx.doi.org/10.1007/s00199-009-0458-9.

Roy Radner and Robert W. Rosenthal. Private information and pure-strategy equilibria. *Math. Oper. Res.*, 7(3):401–409, 1982. ISSN 0364-765X. doi: 10.1287/moor.7.3.401. URL http://dx.doi.org/10.1287/moor.7.3.401.

Aldo Rustichini. Mixing on function spaces. *Econom. Theory*, 3(1):183–191, 1993. ISSN 0938-2259. doi: 10.1007/BF01213701. URL http://dx.doi.org/10.1007/BF01213701.

Yeneng Sun. The exact law of large numbers via Fubini extension and characterization of insurable risks. *J. Econom. Theory*, 126(1):31–69, 2006. ISSN 0022-0531. doi: 10.1016/j.jet.2004.10.005. URL http://dx.doi.org/10.1016/j.jet.2004.10.005.

Jianwei Wang and Yongchao Zhang. Purification, saturation and the exact law of large numbers. *Econom. Theory*, 50(3):527–545, 2012. ISSN 0938-2259. doi: 10.1007/s00199-010-0593-3. URL http://dx.doi.org/10.1007/s00199-010-0593-3.

University of Innsbruck - Working Papers in Economics and Statistics
Recent Papers can be accessed on the following webpage:

http://eeecon.uibk.ac.at/wopec/

2013-18 **Michael Greinecker, Konrad Podczeck:** Purification and independence

2013-17 **Loukas Balafoutas, Rudolf Kerschbamer, Martin Kocher, Matthias Sutter:** Revealed distributional preferences: Individuals vs. teams

2013-16 **Simone Gobien, Björn Vollan:** Playing with the social network: Social cohesion in resettled and non-resettled communities in Cambodia

2013-15 **Björn Vollan, Sebastian Prediger, Markus Frölich:** Co-managing common pool resources: Do formal rules have to be adapted to traditional ecological norms?

2013-14 **Björn Vollan, Yexin Zhou, Andreas Landmann, Biliang Hu, Carsten Herrmann-Pillath:** Cooperation under democracy and authoritarian norms

2013-13 **Florian Lindner, Matthias Sutter:** Level-k reasoning and time pressure in the 11-20 money request game *forthcoming in Economics Letters*

2013-12 **Nadja Klein, Thomas Kneib, Stefan Lang:** Bayesian generalized additive models for location, scale and shape for zero-inflated and overdispersed count data

2013-11 **Thomas Stöckl:** Price efficiency and trading behavior in limit order markets with competing insiders *forthcoming in Experimental Economics*

2013-10 **Sebastian Prediger, Björn Vollan, Benedikt Herrmann:** Resource scarcity, spite and cooperation

2013-09 **Andreas Exenberger, Simon Hartmann:** How does institutional change coincide with changes in the quality of life? An exemplary case study

2013-08 **E. Glenn Dutcher, Loukas Balafoutas, Florian Lindner, Dmitry Ryvkin, Matthias Sutter:** Strive to be first or avoid being last: An experiment on relative performance incentives.

2013-07 **Daniela Glätzle-Rützler, Matthias Sutter, Achim Zeileis:** No myopic loss aversion in adolescents? An experimental note

2013-06 **Conrad Kobel, Engelbert Theurl:** Hospital specialisation within a DRG-Framework: The Austrian case

2013-05 **Martin Halla, Mario Lackner, Johann Scharler:** Does the welfare state destroy the family? Evidence from OECD member countries

2013-04 **Thomas Stöckl, Jürgen Huber, Michael Kirchler, Florian Lindner:** Hot hand belief and gambler's fallacy in teams: Evidence from investment experiments

2013-03 **Wolfgang Luhan, Johann Scharler:** Monetary policy, inflation illusion and the Taylor principle: An experimental study

2013-02 **Esther Blanco, Maria Claudia Lopez, James M. Walker:** Tensions between the resource damage and the private benefits of appropriation in the commons

2013-01 **Jakob W. Messner, Achim Zeileis, Jochen Broecker, Georg J. Mayr:** Improved probabilistic wind power forecasts with an inverse power curve transformation and censored regression

2012-27 **Achim Zeileis, Nikolaus Umlauf, Friedrich Leisch:** Flexible generation of e-learning exams in R: Moodle quizzes, OLAT assessments, and beyond

2012-26 **Francisco Campos-Ortiz, Louis Putterman, T.K. Ahn, Loukas Balafoutas, Mongoljin Batsaikhan, Matthias Sutter:** Security of property as a public good: Institutions, socio-political environment and experimental behavior in five countries

2012-25 **Esther Blanco, Maria Claudia Lopez, James M. Walker:** Appropriation in the commons: variations in the opportunity costs of conservation

2012-24 **Edgar C. Merkle, Jinyan Fan, Achim Zeileis:** Testing for measurement invariance with respect to an ordinal variable *forthcoming in Psychometrika*

2012-23 **Lukas Schrott, Martin Gächter, Engelbert Theurl:** Regional development in advanced countries: A within-country application of the Human Development Index for Austria

2012-22 **Glenn Dutcher, Krista Jabs Saral:** Does team telecommuting affect productivity? An experiment

2012-21 **Thomas Windberger, Jesus Crespo Cuaresma, Janette Walde:** Dirty floating and monetary independence in Central and Eastern Europe - The role of structural breaks

2012-20 **Martin Wagner, Achim Zeileis:** Heterogeneity of regional growth in the European Union

2012-19 **Natalia Montinari, Antonio Nicolo, Regine Oexl:** Mediocrity and induced reciprocity

2012-18 **Esther Blanco, Javier Lozano:** Evolutionary success and failure of wildlife conservancy programs

2012-17 **Ronald Peeters, Marc Vorsatz, Markus Walzl:** Beliefs and truth-telling: A laboratory experiment

2012-16 **Alexander Sebald, Markus Walzl:** Optimal contracts based on subjective evaluations and reciprocity

2012-15 **Alexander Sebald, Markus Walzl:** Subjective performance evaluations and reciprocity in principal-agent relations

2012-14 **Elisabeth Christen:** Time zones matter: The impact of distance and time zones on services trade

2012-13 **Elisabeth Christen, Joseph Francois, Bernard Hoekman:** CGE modeling of market access in services

2012-12 **Loukas Balafoutas, Nikos Nikiforakis:** Norm enforcement in the city: A natural field experiment *forthcoming in European Economic Review*

2012-11 **Dominik Erharter:** Credence goods markets, distributional preferences and the role of institutions

2012-10 **Nikolaus Umlauf, Daniel Adler, Thomas Kneib, Stefan Lang, Achim Zeileis:** Structured additive regression models: An R interface to BayesX

2012-09 **Achim Zeileis, Christoph Leitner, Kurt Hornik:** History repeating: Spain beats Germany in the EURO 2012 Final

2012-08 **Loukas Balafoutas, Glenn Dutcher, Florian Lindner, Dmitry Ryvkin:** The optimal allocation of prizes in tournaments of heterogeneous agents

2012-07 **Stefan Lang, Nikolaus Umlauf, Peter Wechselberger, Kenneth Harttgen, Thomas Kneib:** Multilevel structured additive regression

2012-06 **Elisabeth Waldmann, Thomas Kneib, Yu Ryan Yu, Stefan Lang:** Bayesian semiparametric additive quantile regression

2012-05 **Eric Mayer, Sebastian Rueth, Johann Scharler:** Government debt, inflation dynamics and the transmission of fiscal policy shocks *forthcoming in Economic Modelling*

2012-04 **Markus Leibrecht, Johann Scharler:** Government size and business cycle volatility; How important are credit constraints? *forthcoming in Economica*

2012-03 **Uwe Dulleck, David Johnston, Rudolf Kerschbamer, Matthias Sutter:** The good, the bad and the naive: Do fair prices signal good types or do they induce good behaviour?

2012-02 **Martin G. Kocher, Wolfgang J. Luhan, Matthias Sutter:** Testing a forgotten aspect of Akerlof's gift exchange hypothesis: Relational contracts with individual and uniform wages

2012-01 **Loukas Balafoutas, Florian Lindner, Matthias Sutter:** Sabotage in tournaments: Evidence from a natural experiment *published in Kyklos*

University of Innsbruck

Working Papers in Economics and Statistics

2013-18

Michael Greinecker, Konrad Podczeck

Purification and independence

Abstract
We show that concepts introduced by Aumann more than thirty years ago throw a new light on purification in games with extremely dispersed private information. We show that one can embed payoff-irrelevant randomization devices in the private information of players and use these randomization devices to implement mixed strategies as deterministic functions of the private information. This approach gives rise to very short, elementary, and intuitive proofs for a number of purification results that previously required sophisticated methods from functional analysis or nonstandard analysis. We use our methods to prove a general purification theorem for games with private information in which a player's payoffs can depend in arbitrary ways on events in the private information of other players and in which we allow for shared information in a general way.

ISSN 1993-4378 (Print)
ISSN 1993-6885 (Online)