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Network Formation With Endogenous Decay.[1]

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Abstract

This paper considers a model of economic network characterized by an endogenous architecture and frictions in the relations among agents as described in Bala and Goyal (2000). We propose a similar network model with the difference that frictions in the relations among agents are endogenous. Frictions are modeled as dependent on the result of a coordination game, played by every pair of directly linked agents and characterized by 2 equilibria: one efficient and the other risk dominant. The model has a multiplicity of equilibria and we produce a characterization of those are stochastically stable.

Keywords: Network, Decay, Strategical interaction

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1. Introduction

The network of interactions among socio-economic agents may play an important role for the stability and efficiency of socio-economic systems and a such, it is essential to have theories about how such interaction structures form. Indeed many authors have examined the evolution of interaction in different economic contests. A very common characteristic of networks is the presence of decay, which is the value an individual receives from another as a decreasing function of their distance in the network. Decay could be considered as the effect of generic frictions in the relations among agents, for example noise or delay, which are inevitable in the real world. Generally, network models consider decay as an exogenous characteristic; but we note that in the real word there are many examples where the level of decay depends on some action, decision or behavior of the agents implied in the network. For example, the rate of decay in a communication network could depend on the quality of the device (or technology) used by each agent or by the level of the supplied individual effort. So, in a first step toward reality, we propose a first model of network formation where the rate of decay is endogenously determined by some action chosen by each agent. By the introduction of endogenous decay, the model becomes very complicated and it is very difficult both to provide a complete description of all possible Nash equilibria and to analyze the process of network formation. But using the assumption that agents make mistakes, we provide an almost full description of stochastically stable states without the need to know all possible equilibrium states when agents do not make errors.

We consider the two-way flow network with decay described in Bala and Goyal [1]. They consider a setting in which agents unilaterally[3] form (costly) links in order to access the benefits generated by other agents. These benefits flow in both directions, irrespective of who bears the cost of the link. The benefit that two agents derive from a link depends on the associated level of decay. Their main evidence is that equilibrium network architectures strictly depend on the relation among decay and link cost. Differently from Bala and Goyal [1] we assume that the rate of decay in the network is endogenous: in a given link the rate of decay depends on the results of a social game between the two (directly) linked agents. There are two possible actions: the first one (the efficient action) produces a zero decay if both agents choose it and a maximum decay if the opponent chooses the other action; the second one (the risk dominant action) produces an intermediate value of decay indifferently from the partner’s choice. In this way we model a trade off between

[3] One agent does not need another agent's permission to form a link with him and the cost of link formation is supported only by the agent who initiates the link.

complexity (efficiency) and compatibility (risk dominance).[4] In this way the network structure depends on the set of actions chosen by the agents through the (so determined) level of decay. A first result is that, provided the cost of link formation is not too high, the system converges in states characterized by connected networks and in which agents are coordinated on the same action. When the link's cost is high the system could converge either in states characterized by an empty network or in states with a connected network and in which agents coordinate on the risk dominant action. Moreover, even though we are not able to provide a full description of all equilibrium states the system could converge to, we are able to describe, for a sufficiently large number of agents, stochastically stable states. We find that, for relatively low link cost, the force driving the equilibrium selection in the long run is a trade off between efficiency and risk dominance in the social game. Indeed, stochastically stable states are characterized by agents coordinated on the efficient action if the decay's difference between efficient and risk dominant actions is large enough, otherwise stochastically stable states are characterized by agents coordinated on the risk dominant action. For high cost of link formation, stochastically stable states are characterized either by empty networks or by agents coordinated on the risk dominant action.

Network formation in the presence of decay is studied by, among others, Hojman and Szeidl [12], Watts [25] and Feri [7]. Hojman and Szeidl study a network similar to the two-way flow model with decay described in Bala and Goyal [1], with the difference that agents have concave benefits from connections and decay is modeled in a more general way. Watts considers the dynamics of network formation in the case of the connection model of Jackson and Wolinsky [17] and shows that the resulting network structures are path-dependent. However, this second approach differs significantly from the first one mainly because it restricts attention to network models where the consent of both agents is necessary to form a link[5]. Feri [7] is the most related paper; it considers the two-way flow network with decay described in Bala and Goyal [1] and finds a suitable way to characterize stochastically stable states even in the absence of a full characterization of the equilibria the dynamic process could converge to. Applying this technique we are able to analyze a very complex environment characterized by endogenous decay and endogenous networks. A common result of the above papers is that, given the link cost, the equilibrium architectures strictly depend on the level of decay. However these papers, differently from the present one, study settings where the level of decay is exogenously determined.

[4] This trade off is illustrated by the following example: an individual has to pass a message and can write it in word format or ascii format. The first choice is more efficient only if the reader has the Word software. The second choice is less efficient but all the people can read it.

[5] The link cost is supported by both agents

Strictly related papers are also those of Goyal and Vega-Redondo [10], Hojman and Szeidl [13], Jackson and Watts [15]. Indeed these papers study the interaction between link formation and action choice in a social game. Goyal and Vega-Redondo [10] use a framework where link formation is one side, links are two way-flow and agents interact only with direct neighbors. They find that when the cost of link formation is below a certain threshold then agents coordinate on the risk dominant action, while if the cost is above this threshold agents coordinate on the efficient action. As in our model the social game is played only among directly linked agents but its result affects only the two implied agents. Hojman and Szeidl [13] study a setting where agents interact with direct and indirect neighbours and links are one way flow. They find that long-run equilibrium depends on a trade off between efficiency and risk dominance in the social game. Jackson and Watts [15] study a setting where the consent of both agents is necessary to form a link. Differently from our paper, in these models all equilibrium states, in which network is not empty, are characterized by network architectures that do not depend on which action agents coordinate. This difference is due to the fact that while in these approaches agents establish links to play a coordination game, in our model agents form links to obtain some benefit from other agents and play a coordination game to determine the quality of these benefits; so the result of a single game, determining the decay level in a link, affects the payoff of all agents that use that link in their indirect connections[6].

The paper is organized in the following way: In section 2 we describe the model. Section 3 contains the main result. Section 4 concludes the discussion and provides possible directions for further research.

2. The Model

2.1 Networks

Let $N = \{1, 2, ..., n\}$ be a set of agents where $n \geq 3$. Each agent can obtain some benefit when he is directly or indirectly linked with other agents. Benefits can derive from the transmission of private information held by agents. Although other interpretations are possible we focus on information transmission for simplicity.[7] Without loss of generality, in the following we assume that every agent is endowed with one unit of private information of value 1 as well as of a quantity of information derived from other agents in the network.

Each agent can choose a subset of other agents with whom to establish links. Let $g_i = \left(g_{i,1}, ..g_{i,i-1}, g_{i,i+1}, ...g_{i,n}\right)$ be the set of links formed by agent i where $g_{ij} \in \{0,1\}$ for each

[6] and not only the payoffs of the implied agents.

[7] See Bala and Goyal [1], they use the example of gains from information sharing as source of benefits.

$j \in N \setminus \{i\}$. We say agent *i* forms a link with agent *j* if $g_{ij} = 1$. The set of all agents' link decisions, denoted by $g = (g_1, g_2, ...g_n)$, defines a directed graph called network. With abuse of notation the network will be denoted by *g* and the set of all possible networks will be denoted by *G*. Specifically, the network $g \equiv \{N, \Gamma\}$ has the set of agents N as its set of vertices while its set of directed edges, $\Gamma \subseteq N \times N$, is defined as follows: $\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$. Moreover a sub-network $\mathrm{g}' \subseteq \mathrm{g}$ is a network $\{M, \Gamma_M\}$ where $M \subseteq N$ and $\Gamma_M = \{(i, j) \in M \times M : g_{ij} = 1\}$.

Given a network *g*, we say that 2 agents are directly linked if at least one of them has established a link with the other one, i.e. $\max\{g_{ji}, g_{ij}\} = 1$. To describe the direct links with no regard to who supports them, we define the *closure* $\overline{g}_{ij} = \max\{g_{ij}, g_{ji}\}$. Let $\overline{g}_i = (\overline{g}_{i,1}, .. \overline{g}_{i,i-1}, \overline{g}_{i,i+1}, ... \overline{g}_{i,n})$ be the set of direct links of agent *i*. Then $\overline{g} = (\overline{g}_1, \overline{g}_2, ...\overline{g}_n)$ describes the graph with no regard to who supports the links.

Let $N^d(i; g) \equiv \{j \in N : g_{i,j} = 1\}$ be the set of agents in network *g* with whom agent *i* has established links, while $v^d(i; g) \equiv |N^d(i; g)|$ is its cardinality. In a similar way, let $N^d(i; \overline{g}) \equiv \{j \in N : \overline{g}_{i,j} = 1\}$ be the set of agents in network *g* with whom agent *i* is connected, while $v^d(i; \overline{g}) \equiv |N^d(i; \overline{g})|$ is its cardinality.

We say there is a *path* in *g* between *i* and *j* if there exists a set of agents $P = \{j_1, j_2 .... j_m\} \in N$ where $j_1 = i$ and $j_m = j$ such that $\overline{g}_{j_1 j_2} = \overline{g}_{j_2 j_3} = ... = \overline{g}_{j_{m-1} j_m} = 1$. In the following we denote a path by t_{ij} and by T_{ij} we denote the set of all paths between agents *i* and *j.*

In *g* the distance between agents *i* and *j*, denoted by $d(i, j; g)$, is defined as the number of links of the shortest path in T_{ij}.[8] A sub-network $\mathrm{g}' \subseteq \mathrm{g}$ is called a *component* of *g* if for all $i, j \in M$, $i \neq j$, there exists a path in g' connecting *i* and *j*, and there does not exist a path between an agent in M and one in $N \setminus M$. A network with only one component is called *connected*.

Given any *g*, the notation $g + ij$ denotes the network obtained with the formation of a new link between agents *i* and *j*; similarly, $g - ij$ refers to the network obtained deleting the link between agents *i* and *j*. A connected network is called *minimally connected* and denoted by g^m if $g^m - ij$ is not connected for $\forall i, j \in N$ characterized by $g_{ij} = 1$; a network is called *essential* if $g_{ij} \cdot g_{ji} = 0$

[8] The shorter path is that with the lower number of direct links; if a path between *i* and *j* does not exist we assume $d(i, j; g) = \infty$.

for $\forall i,j\in N$; *empty* and denoted by g^e if $\overline{g}_{i,j}=0$ for $\forall i,j\in N$; *complete* and denoted by g^c if $\overline{g}_{i,j}=1$ for $\forall i,j\in N$; *star* and denoted by g^s if there exists some $i\in N$ such that $\overline{g}_{i,j}=1$ and $\overline{g}_{k,j}=0$ for all $k,j\in N\setminus\{i\}$ and $j\neq k$; among the star networks we denote by g^{cs} the star with all links supported by the central agent, by g^{ps} the star with all links supported by peripheral agents and by g^{ms} all the intermediate cases. Finally we define the following sets of networks: G^m is the set of all minimally connected networks; G^c is the set of all essential g^c; G^s is the set of all essential g^s; G^{ps} is the set of all essential g^{ps}; G^{cs} is the set of all essential g^{cs}.

Links are costly: every agent pays a cost $k>0$ for each link she supports. In our model link formation is one-sided and non-cooperative: the formation of a link requires only the consent of the supporting agent.

2.2 Decay and social game.

Decay is endogenous. We assume that every pair of directly linked agents plays a 2 x 2 symmetric game in strategic form with a common action set given by $A=\{\alpha,\beta\}$. For each pair of actions $a,a'\in A$, the share of information received by an agent choosing *a* when the partner plays *a'* is denoted by $\delta(a,a')$ and it is given by the following table:

(2.1)

	α	β
α	*1*	*0*
β	*e*	*e*

where $0,5<e<1$.

Then, the quantity of information received by an agent choosing *a* when the partner plays *a'* is given by $\delta(a,a')\cdot x$, where *x* is the information owned by the partner. In the bilateral game there are 2 Nash equilibria in pure strategies: (α,α) and (β,β). The first one is efficient, the second one is risk dominant. Each agent plays the game with all directly linked agents and has to use the same action in all engaged bilateral games.

In the following we indicate agents choosing action *a* by *a*-agents where $a\in\{\alpha,\beta\}$, by N_a we denote the set of all *a*-agents and $n_a=|N_a|$. A subset $S\subseteq N$ is called *a-group*, where $a\in\{\alpha,\beta\}$, if $\forall i\in S$ is an *a-agents* and, for all $i,j\in S$, $i\neq j$, there exists a path in $g'\equiv\{S,\Gamma_S\}$ connecting *i* and *j* and does not exist a direct link between an agent in *S* and one *a-agent* in $N\setminus S$. Moreover we say that an *a-group* is minimally connected if deleting any link in the *a-group* we obtain two *a-groups*.

For a generic agent i the strategy space is identified with $S_i = G_i \; x \; A$, where G_i is the set of possible link decisions and A is the common action space of the underlying bilateral game. We denote by $s_i = (g_i, a_i)$ the strategy played by agent i, where a_i is the chosen action and g_i is the set of established links. The state of the system is denoted by $s = (g, a)$ where g is the set of all agents'link decisions and $a = (a_1, ..., a_n)$ is the set of all chosen actions. The set of all possible states is denoted by S.

Now we can present the payoff of the game. Given the strategies of other agents, $s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ...s_n)$, the payoff of agent i deriving from her participation to the game playing some strategy $s_i = (g_i, a_i)$ is given by:

$$\Pi(s_i, s_{-i}) = \sum_{j \in N_i} \left[\prod_{l,k \in \bar{t}_{ij}} \delta(a_l, a_k) \right] - k \cdot v^d(i;g) \qquad (2.2)$$

where $N_i = \{ j : T_{ij} \neq \varnothing \}$ and $\bar{t}_{ij} = arg\,max_{t_{ij} \in T_{ij}} \prod_{l,k \in t_{ij}} \delta(a_l, a_k)$.

These payoff expressions allows us to particularize the standard notion of Nash equilibrium. Thus a state $s^* = (s_1^*, ...s_n^*)$ is said to be a Nash equilibrium if, for all $i \in N$,

$$\Pi(s_i^*, s_{-i}^*) \geq \Pi(s_i, s_{-i}^*) \;\; \forall s_i \in S_i \qquad (2.3)$$

On the other hand a Nash equilibrium will be called strict if every agent gets a strictly higher payoff with her current strategy than she would with any other strategy.

2.3 Dynamics

Time is modelled discretely, and denoted by $t = 1, 2, 3, ...$ At each t, the state of the system is given by strategy profile $s(t) = [g(t), a(t)]$ specifying the strategy $s_i(t) = [g_i(t), a_i(t)]$ for $\forall i \in N$. At every period t one agent is randomly chosen to revise her strategy. When an agent receives this opportunity, she selects a best response to strategy profile in the previous periods:

$$s_i(t) \in \arg\max_{s_i \in S} \Pi[s_i, s_{-i}(t-1)]; \qquad (2.4)$$

If there are several best responses, then any one of them is chosen with equal probability. This strategy revision process defines a Markov chain on $S \equiv S_1 \times S_2 \times ... \times S_n$. In the following we denote this process by *unperturbed dynamics or selection mechanism.*

A no empty set of states $A \subseteq S$ is called *absorbing* if it is a minimal set with respect to the property of being closed under the selection mechanism; hence there is zero probability to transit from a state $s \in A$ to another state $s' \in S \setminus A$.[9] So the possible states the selection mechanism will converge are described by the states contained in absorbing sets . In the following we denote by $\overline{S}$ the set of all states belonging to absorbing sets and by S the set of absorbing sets. As we will see, in our framework, this Markov chain could be characterized by several absorbing sets; in that case which states the selection mechanism will converge, depends upon the initial conditions which in turn motivates the following equilibrium selection.

To select among all possible absorbing sets, we employ the standard techniques used by Kandory, Mailath and Rob [19] and Young [26]. We suppose, conditional on the chance to revise her strategy, agents make mistakes (or mutations). In this case, agent chooses her strategy at random with some small probability $\varepsilon > 0$. For any $\varepsilon > 0$, the process defines an aperiodic and irreducible Markov chain that has a unique invariant probability distribution μ_ε. We analyze the structure of μ_ε as the probability of mistakes ε converges to zero. A state s is called *stochastically stable* if $\hat{\mu}(s) > 0$ where $\hat{\mu} = \lim_{\varepsilon \to 0} \mu_\varepsilon$ and the set of all stochastically stable states is defined as $\hat{S} \equiv \{ s : \hat{\mu}(s) > 0 \}$.

3. Results

In this section we characterize the efficient states, study the characteristics of static equilibria, and analyze the dynamics.

3.1 Efficiency

We use the utilitarian concept of efficiency: the efficient state is that producing the higher total net payoff (gross payoff less cost of links).

Proposition 1*: If $k \le n$, in all efficient states $a_i = \alpha \ \forall i \in N$ and $g \in G^m$. If $k > n$, only states where $g = g^e$ are efficient.*

As the intuition provided below is simple, a formal proof is omitted. Any state with all (or some) agents coordinated on action β is dominated by a state characterized by an equal network and all agents coordinated on action α. When all agents are coordinated on α, from Proposition 4.3 in Bala and Goyal [1] follows that, if $k \le n$, minimally connected networks are efficient otherwise, if $k > n$, the efficient networks are empty.

[9] Moreover we note that an absorbing set may contain many states or may contain only a single state, in this case we call it *absorbing state*.

There are several efficient network architectures: all kinds of star, the line and, more in general, all minimally connected architectures (i.e. with minimum number of links): to connect n agents are necessary at least *n-1* links. Indeed, when all agents are coordinated on action α, there are not differences in payoff between a direct link and an indirect one. If $\delta(\alpha,\alpha)<1$, we could restrict the set of efficient networks because the share of information arriving from an agent to another is decreasing with the number of links that has to pass through. In accordance with Proposition 1 in Jackson and Wolinsky [17], we find that the efficient networks are either complete or stars or empty depending on the link cost k.

3.2 Static equilibria.

Our first result concerns the nature of states that arise in equilibrium. In this setting we have a multiplicity of Nash equilibria, such as, among others, states characterized by agents choosing different actions. A Nash equilibrium in which some agent has multiple best responses is likely to be unstable since this agent can decide to switch to another payoff-equivalent strategy. This motivates an examination of strict Nash equlibria that are described in the following proposition.

Proposition 2: *Let $s=(g,a)$ be a strict Nash equilibrium. Then g is essential, connected and $a_i=a_j\ \forall i,j\in N$. Moreover: a) if $k>1$ $a_i=\beta\ \forall i\in N$ b) if $a_i=\alpha\ \forall i\in N$ then $g\in G^{cs}$.*

The Proposition describes two important features of strict Nash equilibria: aggregation and conformity. A state characterized by a network with two or more separated components is not a strict Nash equilibrium because always some agent prefers to supports links with no connected agents. The special case of a state characterized by empty network cannot be strict because, even if the incentives to form links are not enough, agents indifferently choose an action or another. A state with agents coordinated on different actions cannot be strict because β-agents are indifferent on which α-agent to be linked. Finally we note that only for $k<1$ efficient states are strict Nash equilibria. Indeed, in this range of link cost, states where all agents are coordinated on efficient action could be strict Nash equilibria as well as states where all agents are coordinated on the risk-dominant one; in the first case the proposition says that the network has to be a center-sponsored star; for $k>1$ only states where all agents are coordinated on the risk-dominant action can be strict Nash.

Proof. The proof goes in two steps. In the first one we prove that in all strict Nash equilibria where $a_i=a_j\ \forall i,j\in N$ networks are essential and connected. In the second step we show that a state where $\exists i,j\in N$ such that $a_i\neq a_j$ is never a strict Nash equilibrium. <u>Step 1</u>. Assume a strict Nash equilibrium where $a_i=\alpha\ \forall i\in N$. From Proposition 4.2 in Bala and Goyal [1] we know that when $k<1$ only a state characterized by a $g \in G^{cs}$ is a strict Nash equilibrium. In this state every agent obtains a strictly positive payoff. Changing action an agent could obtain a payoff reduction of at least of $1-e$ for each connected agent. Then the considered state is a strict Nash equilibrium. For $k>1$, from Proposition 4.2 in Bala and Goyal [1] we know that the unique candidate to be a strict Nash equilibrium is any state characterized by an empty network. But in our model these states are never strict Nash because agents can switch action obtaining the same (zero) payoff. Now, assume a strict Nash equilibrium where $a_i = \beta \ \forall i \in N$. The proof for essentiality and connecteness derives directly from Proposition 5.3 in Bala and Goyal [1]. If an agent switches to action α, she obtains a zero payoff. Then the considered state is a strict Nash equilibrium. Step 2. Consider any strategy profile where n_α agents are choosing action α, n_β agents are choosing action β and $n_\alpha > 1$.[10] Using the same arguments as in Proposition 4.2 in Bala and Goyal [1] we know that, if k<1, in a strict Nash equilibrium all α-agents have to link among themselves in a g^{cs}. Suppose that β-agents are in one or more separated components. This state cannot be a strict Nash equilibrium because any β-agent, forming a link with an α-agent, could obtain an additional payoff of $n_\alpha \cdot e - k$ that is strictly positive. All connected states where α-agents support links with β-agents cannot be strict Nash equilibrium because α-agents obtain a negative net payoff from their links with β-agents [11]. Finally, all connected states where β-agents support links with α-agents cannot be strict Nash equilibrium because β-agents are indifferent on which α-agent to be tied. In the case for $k \geq 1$, in a strict Nash equilibrium all agents must be β-agents because a strict Nash equilibrium for α-agents does not exist. QED.

3.3 Dynamics

In this section we describe the dynamic properties of different equilibria. First we describe the characteristics of states where the selection mechanism will converge. Then we provide an almost complete description of stochastically stable states.

The following proposition give us a description of the states belonging to absorbing sets.

Proposition 3:

I) Let $k < e - e^2$. *Then* $\overline{S} = \left\{ s \in S : \left(g \in G^m \wedge a_i = \alpha \ \forall i \in N \right) \vee \left(g \in G^c \wedge a_i = \beta \ \forall i \in N \right) \right\}$.

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[10] The special case with $n_\alpha = 1$ cannot be a strict Nash equilibrium because the unique α-agent obtains zero payoff: she could change action obtaining at least a zero payoff (for example, if she changes action and does not link with anyone, she obtains zero payoff).

[11] an α-agent supporting one link with a β-agent obtains a negative payoff of *-k*.

II) Let $e-e^2<k<1$. *There exists a* $\overline{n}(k,e)$ *such that for all* $n>\overline{n}(k,e)$ *if* $s\in\overline{S}$ *either* $a_i=\beta\ \forall i\in N$ *and g is essential and connected or* $a_i=\alpha\ \forall i\in N$ *and* $g\in G^m$.

III) Let $1<k<e+(n-2)\cdot e^2$. *If* $s\in\overline{S}$ *either* $g=g^e$ *or g is essential and connected and* $a_i=\beta\ \forall i\in N$.

IV) Let $k>e+(n-2)\cdot e^2$. *Then* $\overline{S}=\{s\in S: g=g^e\}$

Proof. To show that only states described in the proposition can belong to absorbing sets it is enough to prove that under the selection mechanism: *a)* from any $s\in S$ there is a strictly positive probability to go in one state described in the proposition; *b)* there is zero probability to move from the states described in proposition to states that are not. We need the following lemmas.

Lemma 1: *From any state where* $a_i=\alpha\ \forall i\in N$, *the selection mechanism goes, with probability 1, in a state where:*

a) $a_i=\alpha\ \forall i\in N$ *and* $g\in G^m$ *if* $k\leq 1$

b) $g=g^e$ *if* $k>1$

The proof is in the appendix.

Lemma 2: *From any state where* $a_i=\beta\ \forall i\in N$, *the selection mechanism goes, with probability 1, in states where:*

a) $a_i=\beta\ \forall i\in N$ *and* $g\in G^c$, *if* $k\leq e-e^2$.

b) $a_i=\beta\ \forall i\in N$ *and g is essential and connected, if* $e-e^2<k\leq e$.

c) $a_i=a_j\ \forall i,j\in N$ *and g is essential and connected network, if* $e<k\leq 1$.

d) either $a_i=\beta\ \forall i\in N$ *and g is essential and connected or* $g=g^e$, *if* $1<k\leq e+(n-2)\cdot e^2$.

e) $g=g^e$ *if* $k>e+(n-2)\cdot e^2$

The proof is in the appendix.

Now we can prove proposition 3. Consider any state where $a_i=a_j\ \forall i,j\in N$. The results in Lemma 1 and 2 are enough to prove the convergence to states with the characteristics described in proposition. Now consider states where $n_\alpha>0$ and $n_\beta>0$.

<u>Part *I*</u> ($k\leq e-e^2$). Candidates to be absorbing states are those characterized by an unique and

completely connected β-group, with all β-agents supporting one link to the unique and minimally connected α-group. This result derives from a slight variation of part *a* of Lemma 1, part *a* of Lemma 2 and from the consideration that β-agents receive from one link with the α-group, a payoff equal to $n_\alpha e - k > 0$. A direct link with the α-group is preferred with respect to an indirect one because $n_\alpha e - k > n_\alpha e^2$. We note that β-agents are indifferent on which α-agent to be linked and any α-agent is chosen with equal probability. Then there is a strictly positive probability that the dynamic process goes in a state where all β-agents are linked to the same α-agent. Denote such agent by *i* and suppose that she is supporting *x* links. Her payoff is $\Pi_i(\alpha) = (n_\alpha - 1) - x \cdot k$, while switching to action β she obtains $\Pi_i(\beta) = (n_\alpha + n_\beta - 1) \cdot e - x \cdot k$. The condition such that agent *i* does not switch action is $\Pi_i(\alpha) > \Pi_i(\beta)$, which can be rewritten as:

$$n_\alpha > \frac{n_\beta \cdot e}{1 - e} + 1 \tag{3.1}$$

Now consider a β-agent and denote him by *j*. Suppose that agent *j* is supporting *x* links, of which *x-1* with β-agents and one link with the α-group. Her payoff is $\Pi_j(\beta) = (n_\alpha + n_\beta - 1) \cdot e - x \cdot k$, while switching to action α she obtains $\Pi_j(\alpha) = n_\alpha - k$. The condition such that agent *j* does not switch action is $\Pi_j(\beta) > \Pi_j(\alpha)$, that can be rewritten as:

$$n_\alpha < \frac{(n_\beta - 1) \cdot e - (x - 1) \cdot k}{1 - e} \tag{3.2}$$

We note that it is impossible to satisfy both conditions (3.1) and (3.2); indeed the right part of (3.1) is larger that right part of (3.2) for any value $x \geq 1$. It follows that at most one of these two conditions can be satisfied. Suppose the first one is not satisfied; if given the chance to revise, α-agents switch to action β; suppose that (3.2) is not satisfied; if given the chance to revise β-agents switch to action α. Then, by a strictly positive probability, the selection mechanism goes in a state where $a_i = a_j \forall i, j \in N$. The results described in Lemma 1 and 2 are enough to complete the proof of the convergence to states described in the proposition. Finally we show there is zero probability to move from the states described in part I of the proposition to states that are not. Consider any state characterized by $a_i = \beta \ \forall i \in N$ and $g \in G^c$. The selection mechanism cannot move in another state, indeed it is directly verifiable that *a)* to change action is not a best response because it produces a zero payoff; *b)* all changes in the link strategy are not best responses because they reduce the payoff. Consider any state characterized by $a_i = \alpha \ \forall i \in N$ and $g \in G^m$. The selection mechanism cannot move in another kind of state, indeed it is directly verifiable that to change action as well as a change in the link strategy such that the resulting network $g' \notin G^m$, are not best response because they cause a payoff reduction.

Part *II* ($e - e^2 < k < 1$). Candidates to be absorbing states are those characterized by an unique and connected *β*-group, with some *β*-agent supporting one link to the unique and minimally connected *α*-group. This result derives from a slight variation of part *a* in Lemma 1, part *b* of Lemma 2 and from the consideration that *β*-agents receive, from a link with the *α*-group, a payoff equal to $n_\alpha e - k$. Now we show that there exists a $\bar{n}(k,e)$ such that for all $n > \bar{n}(k,e)$ all candidate states to be absorbing are characterized by $n_\alpha > k/(e-e^2)$. Denote by *i* any β-agent directly connected with the α-group. Her payoff is $\Pi_i(\beta) = f_i(g, n_\beta) + n_\alpha \cdot e - k$ where $f_i(g, n_\beta)$ is the net payoff accruing from the β-group. Switching to action α agent *i* obtains $\Pi_i(\alpha) = n_\alpha - k$. The condition such that agent *i* does not switch action is $\Pi_i(\beta) > \Pi_i(\alpha)$, that rewritten is:

$$n_\alpha < \frac{f_i(g, n_\beta)}{1-e} \tag{3.3}$$

Now denote by *j* any α-agent supporting *x* links and receiving all links that β-agents are supporting with the α-group. Her payoff is $\Pi_j(\alpha) = (n_\alpha - 1) - x \cdot k$; switching to action β, agent *j* obtains $\Pi_j(\beta) = f_j(g, n_\beta) + (n_\alpha - 1) \cdot e - x \cdot k$ where $f_j(g, n_\beta)$ is the net payoff accruing from the β-group. Agent *j* does not switch action if $\Pi_i(\alpha) > \Pi_i(\beta)$ that rewritten is:

$$n_\alpha > 1 + \frac{f_j(g, n_\beta)}{1-e} \tag{3.4}$$

Note that increasing the number of agents, both n_α and n_β have to increase. Indeed increasing only n_α equation (3.3) will become unsatisfied, as well as increasing only n_β, equation (3.4) will become unsatisfied [12]. Then it exists a $\bar{n}(k,e)$ such that for all $n > \bar{n}(k,e)$ to satisfy condition (3.4) must be $n_\alpha > k/(e-e^2)$. So when *n* is large enough, candidates to be absorbing states are those characterized by an unique and connected β-group, with each β-agent supporting one link to the unique and minimally connected α-group. Given that β-agents are indifferent on which α-agent to

[12] Given that agent *j* is receiving all links that *β*-agents support to the *α*-group, the $max\ d(j,z;g)$ $z \in N_\beta$ is given by the smaller integer $\bar{d}$ such that $n_\alpha e - k < n_\alpha e^{\bar{d}}$. Otherwise, if it exists an agent $z \in N_\beta$ such that $d(j,z;g) > \bar{d}$, the state under consideration cannot be absorbing because agent *z* can profitably change strategy supporting a direct link with the *α*-group. Therefore, increasing the number of β-agents, $f_j(g, n_\beta)$ has to increase at lest of $e^{\bar{d}}$ for each new β-agent.

be linked, there exists a positive probability that dynamic process goes in a state where all β-agents are linked to the same α-agent. The condition such that an α-agent, receiving links from all β-agents, does not switch action is equal to (3.1). Moreover consider the condition (3.3) such that a β-agent does not switch to action α; the less strict condition is for $f_j(g,n_\beta)=(n_\beta-1)\cdot e$ and it is equal to (3.2) with $x=1$. We note that it is impossible to satisfy both conditions; then by a strictly positive probability the selection mechanism goes in a state where $a_i = a_j \forall ij \in N$. The results in Lemma 1 and Lemma 2 are enough to complete the proof of the convergence to states described in the proposition. The rest of the proof uses very similar arguments than in part *I* and it is omitted.

<u>Part *III*</u> ($1 < k < e + (n-2)\cdot e^2$). Giving repeatedly the chance to revise the strategy only to α-agents, they delete their links between them. The proof of this result is omitted because use similar arguments than in Theorem 4.1 in Bala and Goyal and in Lemma 1 part b. When an α-agent deletes all her links remaining no connected with the other α-agents, she switches action with some positive probability. So, there is a positive probability to go in a state where $a_i = \beta \forall i \in N$. Then, the result stated in Lemma 2, part d it is enough to complete the proof.

<u>*Part IV*</u> ($k > e + (n-2)\cdot e^2$). This result derives from Lemma 1, part b and Lemma 2 part *e*. QED.

For any interval of link cost the proposition describes more than one absorbing set. When $k<1$ we can partition states belonging to absorbing sets into those in which agents coordinate on action α and those in which agents coordinate on action β; if $k>1$ we can divide states with agents coordinated on action β from those characterized by empty network (without interaction between agents).[13] We note that emerging network structures depend on the action on which agents are coordinated. So, given the multiplicity of absorbing sets, the final outcome of the dynamic process depends on initial conditions. Natural questions are about the most probable states to emerge in this process or which are the most "robust" states to perturbations, inevitable in the real word. But to answer these questions a complication is that we are not able to produce a full description of states belonging to absorbing sets in the interval $e - e^2 < k < e + (n-2)\cdot e^2$. In spite of all that, we provide an almost complete description of a special subset of absorbing sets using the concept of stochastic stability. To do this selection the result stated in proposition 3 is important because it delimits the

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[13] Moreover, among the states described in the proposition we can identify some important class of states. For example, among states $s \in \overline{S}$ such that $a_i = \alpha \forall i \in N$ we find that those characterized by $g \in G^{cs}$ are absorbing states13; moreover among states $s \in \overline{S}$ such that $a_i = \beta \forall i \in N$, when $e - e^2 < k < e$ we find that those characterized by $g \in G^{s}$ are absorbing, as well as states characterized by $g \in G^{ps}$ when $e < k < e + (n-2)\cdot e^2$.

set of states that can potentially be stochastically stable since every such state must be a limit point for the unperturbed dynamics. The following theorem summarizes our analysis.

Theorem 1: *There exists* $\hat{n}$ *such that for all* $n > \hat{n}$ *:*

I) *Let* $k < e - e^2$. *If* $k > 2 - \frac{1}{e}$ *then* $\hat{S} = \left\{ s \in \overline{S} : a_i = \alpha \ \forall i \in N \right\}$*; otherwise* $\hat{S} = \left\{ s \in \overline{S} : a_i = \beta \ \forall i \in N \right\}$.

II) *Let* $e - e^2 < k < e$. *If* $k > \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$ *then* $\hat{S} = \left\{ s \in \overline{S} : a_i = \alpha \ \forall i \in N \right\}$*; otherwise* $s \in \hat{S}$ *implies* $s \in \overline{S}$ *such that* $a_i = \beta \ \forall i \in N$ *and* $\hat{S} \supseteq \left\{ s \in \overline{S} : g \in G^s \wedge a_i = \beta \ \forall i \in N \right\}$*, with equality if* $k > e - e^3$.

III) *Let* $e < k < 1$. $\hat{S} \supseteq \left\{ s \in \overline{S} : a_i = \alpha \ \forall i \in N \right\}$*, with equality if* $k < 1 - e^2 - \frac{e}{n-2}$.

IV) *Let* $k > 1$. $\hat{S} \supseteq \left\{ s \in \overline{S} : g^e \right\}$*, with equality if* $k > \frac{e + (n-1) \cdot e^2}{2}$.

In order to determine which states are stochastically stable we use the techniques introduced by Kandory, Mailath and Rob [19] and Young [26] that can be summarized as follows. Fix some absorbing set of states $A' \in S$. An *A-tree* is a directed graph on S whose root is *A* and such that there is a unique (directed) path joining any other absorbing set $A' \in S$ to *A*. For each arrow *A' →A"* in any given *A-tree*, a cost is defined as the minimum number of mutations that are required for the transition from *A'* to *A"* to be feasible through the ensuing operations of the unperturbed dynamics alone. The cost of the tree is obtained by adding up the costs associated with all the arrows of a particular *A-tree*. The stochastic potential of an absorbing set *A* is defined as the minimum cost across all *A-trees*. Then an absorbing set *A* is stochastically stable if it has the lowest stochastic potential across all absorbing sets.

To prove this Theorem we use the notion of recurrent set in the sense of Definition 7.4 in Samuelson [22]: a recurrent set $X \subseteq S$ is a collection of absorbing sets with the following two properties: *a*) it is impossible starting from an absorbing set $A \in X$, to end up in an absorbing set $A' \notin X$, by means of a single perturbation followed by the selection mechanism; *b*) given two absorbing sets $A', A'' \in X$, we can find a sequence of absorbing sets in X , $A_1 .... A_m .... A_M$ with $A_1 = A'$ and $A_M = A''$, such that for any $m \in [2, M]$ is possible to move from A_{m-1} to A_m by a transition that includes a single mutation followed by unperturbed dynamics. In the following we denote this kind of sequence as *path of one step mutations*. Sometime, in the following, we describe a recurrent set as the set of states contained in the various absorbing sets that constitute the recurrent set, that is the union set of all absorbing sets in it.

Finally we use the following results in Samuelson [22]: 1) By Proposition 7.4: a state is stochastically stable only if it is contained in an absorbing set, moreover all other states belonging to the same absorbing set are stochastically stable. 2) By Proposition 7.7: if a state is stochastically stable, then it is contained in a recurrent set and all states in the same recurrent set are stochastically stable. The first result permits us to concentrate our attention only to states with characteristics described in Proposition 3. The second one permits us to simplify the computations to find the set of stochastically stable states: if only one recurrent set exists, all states belonging to it are stochastically stable and we do not need to compute stochastic potential; if two or more recurrent sets exist we have to compute the stochastic potential only for absorbing sets belonging to a recurrent set.

Proof. In the proof we need of the following notation: assume that S_h and $S_{h'}$ are recurrent sets, by $m_{h,h'}$ we denote the minimum (mutation) cost across all paths joining some state in S_h to some state in $S_{h'}$, and with abuse of notation, by $|S_h|$ we denote the number of absorbing sets in S_h; finally, by $\lceil z \rceil$ we denote the smallest integer no smaller that any given $z \in R^+$. We note that, by Proposition 7.4 in Samuelson [22], candidates to be stochastically stable are the states described in Proposition 3. In the proof we identify the recurrent sets and, if two or more recurrent sets exist, we prove which is stochastically stable.

Part *I* ($k \le e - e^2$). By the result stated in proposition 3 we identify two candidates to be recurrent sets: $S_\beta = \{ s \in S : g \in G^c \wedge a_i = \beta \ \forall i \in N \}$ and $S_\alpha = \{ s \in S : g \in G^m \wedge a_i = \alpha \ \forall i \in N \}$. We need the following lemmas:

Lemma 3: *Let $k \le 1$. For any pair of absorbing sets $A', A'' \subset S_\alpha$, there exists a path of one-step mutations in S_α, that leads from A' to A''.*

The proof is in the appendix.

Lemma 4: *Let $k \le e - e^2$. Then,* $m_{\alpha\beta} = \lceil (n-1)\cdot(1-e) \rceil$ *and* $m_{\beta\alpha} = \left\lceil (n-1)\cdot \frac{(e-k)}{1-k} \right\rceil$.

Moreover, for n sufficiently large:

a) $m_{\alpha\beta}, m_{\beta\alpha} > 1$*;*

b) if $k > 2 - \frac{1}{e}$ *then* $m_{\alpha\beta} - m_{\beta\alpha} > 0$*; otherwise* $m_{\alpha\beta} - m_{\beta\alpha} < 0$.

The proof is in the appendix.

To prove that all $s \in S_\beta$ are in the same recurrent set is sufficient to check property (b) of recurrent sets. The proof is very similar to that of part I of Theorem I in Feri [7] and it is omitted[14]. To prove that all $s \in S_\alpha$ are in the same recurrent set the result in lemma 3, satisfying property (b) of recurrent sets, is sufficient: assume that S_α is split into two or more subsets and each subset is contained in a separate recurrent set. The result stated in Lemma 3 is in contradiction with property (a) of recurrent sets. Therefore only S_α and S_β are candidates to be recurrent sets. The result stated in part a of Lemma 4 is enough to prove that, for n sufficiently large, S_α and S_β are two separate recurrent sets. Indeed this result says us that both S_α and S_β satisfy property (a) for recurrent sets. Given that only states in S_α and S_β are candidates to be stochastically stable we have to compute the stochastic potential only for them. It is directly verifiable that A-trees for any absorbing set $A \subseteq S_\alpha$ will have a minimum cost of $m_{\beta\alpha} + |S_\alpha| + |S_\beta| - 2$, while A-trees for any absorbing set $A \subseteq S_\beta$ will have a minimum cost of $m_{\alpha\beta} + |S_\alpha| + |S_\beta| - 2$. Then, to determine the stochastically stable set of states we simply need to compare $m_{\alpha\beta}$ and $m_{\beta\alpha}$. The rest of the proof derives directly from part b of Lemma 4.

<u>Part *II*</u> ($e - e^2 \le k \le e$). We need of the following lemma.

Lemma 5: *Let* $e - e^3 \le k \le e$ *and suppose any* $s \in \{s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N\}$. *Then exists* $n'(k,\delta)$ *such that after a single mutation followed by unperturbed dynamic, the state converges to any* $s \in \{s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N\}$ *if* $n > n'(k,\delta)$.

The proof is in the appendix.

By the result stated in proposition 3 we identify two candidates to be recurrent sets: $S_\alpha = \{s \in S : g \in G^m \wedge a_i = \alpha \ \forall i \in N\}$ and $S_\beta \subseteq \{s \in \bar{S} : a_i = \beta \ \forall i \in N\}$. By the proof of part I, we know that S_α satisfies property (b) of recurrent sets and that all $s \in S_\alpha$ have to be in the same recurrent set. Now we show that S_β has to contain all states such that $a_i = \beta \ \forall i \in N$ and $g \in G^s$ and that, among all absorbing sets of states such that $a_i = \beta \ \forall i \in N$, only S_β could be a recurrent set. The proof of this statement goes in two steps. In the first we note that a recurrent set

[14] To use the proof in part I of Theorem I in Feri [7] is sufficient to assume mutations in which agents change only the link strategy and continue to use action β.

characterized by states where $a_i = \beta \ \forall i \in N$ but without states characterized by $g \in G^{cs}$ cannot exist. The proof is based on a slight variation of Lemma 1 in Feri[7][15]: assume S_β does not contain states characterized by $g \in G^{cs}$; a single mutation in the link strategy, followed by unperturbed dynamic, is sufficient to move in a state where $a_i = \beta \ \forall i \in N$ and $g \in G^{cs}$. This fact violates property (*a*) of recurrent sets. In the second step we show that all states $s \in S$ such that $a_i = \beta \ \forall i \in N$ and $g \in G^s$ have to be in S_β. The proof is based on a slight variation of Lemma 2 in Feri[7][16] satisfying property (*b*) of recurrent sets: assume that some state $s \in S$ such that $a_i = \beta \ \forall i \in N$ and $g \in G^s$ is not contained in S_β; the result stated in Lemma 2 in Feri [7] is in contradiction with property (*a*) of recurrent sets. These results together tell us that $S_\beta \supseteq \left\{ s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N \right\}$ and, among states characterized by $a_i = \beta \ \forall i \in N$, only S_β could be a recurrent set. Moreover, for n large enough and $e - e^3 \leq k \leq e$, $S_\beta = \left\{ s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N \right\}$. The prove this statement the result in Lemma 5 is sufficient. Therefore only S_α and S_β are candidates to be recurrent sets.

Now we need of the following lemma:

Lemma 6: *Let* $e - e^2 < k < e$. *Then,* $m_{\alpha\beta} = \left\lceil (n-1) \cdot \frac{1 - e^2 - k}{1 + e - e^2 - k} \right\rceil$ *and* $m_{\beta\alpha} = \left\lceil (n-1) \cdot \frac{e-k}{1-k} \right\rceil$.

Moreover, for n sufficiently large:

a) $m_{\alpha\beta}, m_{\beta\alpha} > 1$*;*

b) if $k > \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$ *then* $m_{\alpha\beta} - m_{\beta\alpha} > 0$*; otherwise* $m_{\alpha\beta} - m_{\beta\alpha} < 0$.

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[15] Lemma 1 in Feri [7] can be modified in the following way: *Let* $e - e^2 < k < e$. *To induce a transition from any state such that* $a_i = \beta \ \forall i \in N$ *to a state where* $a_i = \beta \ \forall i \in N$ and $g \in G^{cs}$ *it is sufficient to have one mutation followed by unperturbed dynamic.* The proof of this statement is very similar to that in Feri [7]: it is sufficient to assume an initial mutation in which the "mutant" agent changes only the link strategy and continue to use action β. Given that for all agents to change action is not a best response because produces a zero payoff , it is directly verifiable that the proof is the same.

[16] Lemma 2 in Feri [7] can be modified in the following way: *Let* $e - e^2 < k < e$. *For any pair* $s', s'' \in \left\{ s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N \right\}$, *a path of one-step mutations, that leads from* s' *to* s'', *exists in* $\left\{ s \in S : g \in G^s \wedge a_i = \beta \ \forall i \in N \right\}$. The proof of this statement is very similar to that in Feri [7]: it is sufficient to assume mutations in which agents change only the link strategy and continue to use action β.

The proof is in the appendix.

The results in part *a* of Lemma 6 is enough to prove that, for *n* large enough, S_α and S_β are two separate recurrent sets. Indeed this result says us that both S_α and S_β satisfy property (a) for recurrent sets. Given that only states $s \in \{S_\alpha \cup S_\beta\}$ can be candidates to be stochastically stable we have to compute the stochastic potential only for them. The principal complication is that we are not able to describe all absorbing sets in which $a_i = \beta \ \forall i \in N$ and that could be not belonging to S_β. Denote this set by $A_\beta = \{s \in \overline{S} : a_i = \beta \ \forall i \in N \wedge s \notin S_\beta\}$. We note that one mutation is enough to cause a transition from any absorbing set characterized by states in which $a_i = \beta \ \forall i \in N$ to a state $s \in S$ such that $g \in G^{cs}$ and $a_i = \beta \ \forall i \in N$. This statement derives directly by the slight variation of Lemma 1 in Feri [7] (see footnote 14). It is directly verifiable that A-trees for any absorbing set $A \subseteq S_\alpha$ will have a minimum cost of $m_{\beta\alpha} + |S_\alpha| + |S_\beta| + |A_\beta| - 2$, while A-trees for any $A \subseteq S_\beta$ will have a minimum cost of $m_{\alpha\beta} + |S_\alpha| + |S_\beta| + |A_\beta| - 2$. Therefore to determine the stochastically stable set of states we simply need to compare $m_{\alpha\beta}$ and $m_{\beta\alpha}$. The rest of the proof derives directly from the result stated in part *b* of Lemma 6.

Part *III* ($e \leq k \leq 1$). Let $S_\alpha = \{s \in S : g \in G^m \wedge a_i = \alpha \ \forall i \in N\}$. We need the following lemmas:

Lemma 7: *Let* $e < k < 1$. *To induce a transition from any state* $s \in S$ *such that* $a_i = \beta \ \forall i \in N$ *to a state* $s \in S_\alpha$ *it is sufficient one mutation followed by unperturbed dynamics.*

The proof is in the appendix.

Lemma 8: *Let* $e < k < 1$. *From any* $s \in S_\alpha$, *after a single mutation followed by unperturbed dynamics the system converges to* $s \in S_\alpha$ *if* $k < 1 - e^2 - \dfrac{e}{n-2}$.

The proof is in the appendix.

A recurrent set without $s \in S_\alpha$ cannot exist. To prove this statement Lemma 7 is sufficient: assume there is a recurrent set that does not contain any state $s \in S_\alpha$; a single mutation, followed by an unperturbed dynamics, is sufficient to move the system in a state $s \in S_\alpha$. This fact violates property (*a*) of recurrent sets. By the proof of part I, we know that S_α satisfies property (*b*) of recurrent sets and that all $s \in S_\alpha$ have to be in the same recurrent set. Therefore there exists only one recurrent set containing all $s \in S_\alpha$. If $k < 1 - e^2 - e/(n-2)$, the unique recurrent set contains only $s \in S_\alpha$. To prove this, Lemma 8 is sufficient. Given that only one recurrent set exists, all states belonging to it are stochastically stable (Proposition 7.7 in Samuelson [22]).

Part *IV* ($k \geq 1$). We need the following lemmas:

Lemma 9: *Let $k > 1$. To induce a transition from any state $s \in S$ such that $a_i = \beta\ \forall i \in N$ to a state $s \in S$ such that $g = g^e$ it is sufficient one mutation followed by unperturbed dynamics.*

The proof is in the appendix.

Lemma 10: *Let $k > 1$. From any state $s \in S$ such that $g = g^e$, after a single mutation followed by unperturbed dynamics the state converges to any $s \in \overline{S}$ such that $g = g^e$ if $k > e + \frac{n-1}{2} \cdot e^2$*

The proof is in the appendix.

In the first step we show that recurrent sets without $s \in S$ such that $g = g^e$ cannot exist. To prove this statement Lemma 9 is sufficient: assume there is a recurrent set that does not contain $s \in S$ such that $g = g^e$; a single mutation, followed by an unperturbed dynamics, is sufficient to move the system into $s \in S$ such that $g = g^e$. This fact violates property (a) of recurrent sets. In the second step we show that, for sufficiently large values of *k*, the unique recurrent set contains only $s \in S$ such that $g = g^e$. To prove this, Lemma 10 is sufficient. Given that only one recurrent set exists, all states belonging to it are stochastically stable (Proposition 7.7 in Samuelson [22]). QED

4. Discussion on the result

Ideally we would like to have a complete characterization of stochastically stable states for all values of *k* but we have been unable to describe completely the interaction structure in those characterized by $a_i = \beta \forall i \in N$ when $e - e^2 < k < e - e^3$ and $k > e$. This fact is due to the difficulty to prove Lemma 5 in the interval $e - e^2 < k < e - e^3$ and an equivalent lemma for states $s \in \overline{S}$ such that $a_i = \beta \forall i \in N$ and $g \in G^{ps}$ when $k > e$. [17]

We illustrate the stochastically stable states using the following figure, where we display their main characteristics according to the levels of *e* and *k* (x-axis displays *e* and y-axis displays *k*).

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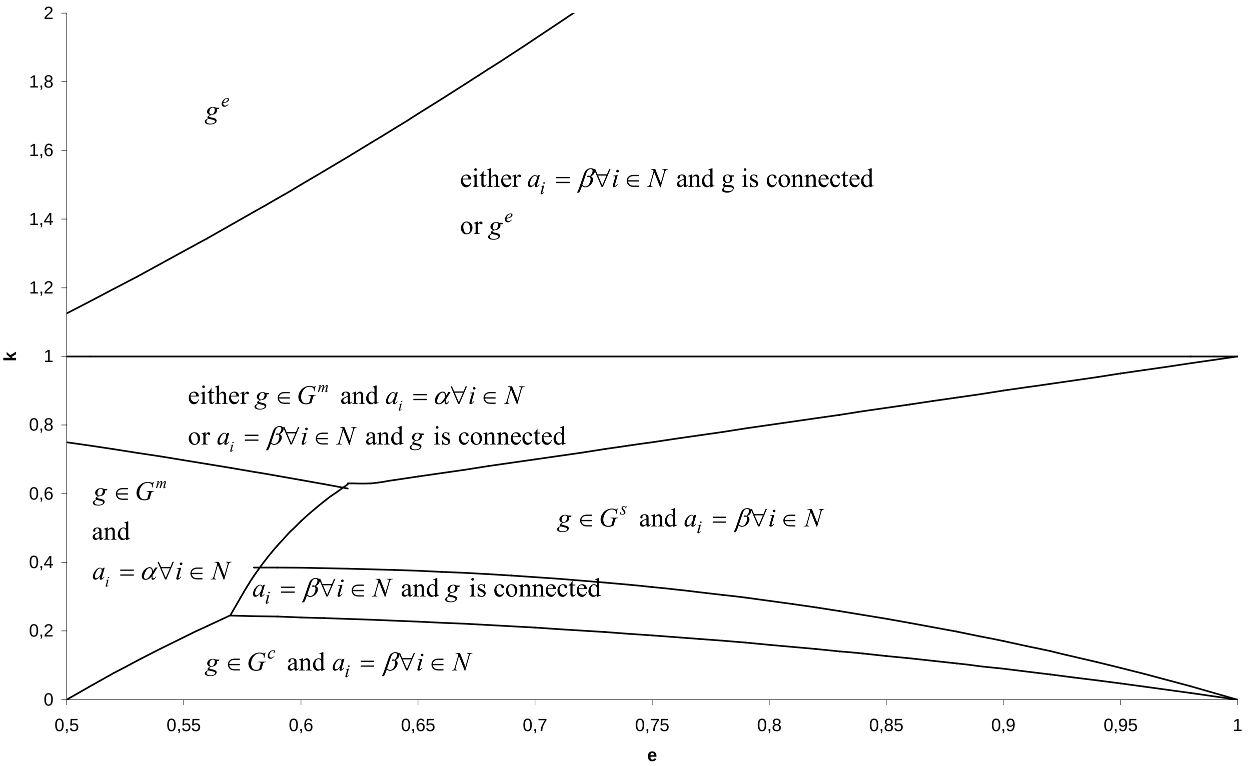
[17] By the proofs of Lemmas 6 and 8 it is directly verifiable that states $s \in \overline{S}$ such that $a_i = \beta \forall i \in N$ and $g \in G^{ps}$ are belonging to the unique recurrent set when $1 - e^2 - \frac{e}{n-2} < k < \frac{e + (n-1) \cdot e^2}{2}$ and *k*>*e*. Moreover we note that, conditional on $a_i = \beta \forall i \in N$, these states are characterized by an efficient interaction structure.



We note that there is a trade off between compatibility and efficiency. When the premium[18] to play efficient action (α) is small, states characterized by coordination on the risk dominant action (β) are stochastically stable because they are more robust to perturbations; otherwise stochastically stable states are those characterized by the efficient action. The intuition is that agents choosing action α receive more payoff from other α-agents but receive nothing from β-agents, while β-agents receive payoff from all agents. So when an agent switches from β to α, her payoff increases of at least $1-e$ for each α-agent but decreases of e for each β-agent; on the contrary, when an agent switches from α to β, her payoff decreases of at least $1-e$ for each α-agent and increases of e for each β-agent. It follows that smaller $1-e$, the more mutations are needed to cause a transition from states characterized by coordination on β action (the risk dominant action) to states in which agents coordinate on action α (the efficient one) and the less mutations are needed for the reverse transition.

Externalities generated by passive links are a second cause that increases robustness of states characterized by coordination on the risk dominant action because they decrease the number of perturbations needed to switch from states with all agents choosing action α to states with all agents choosing action β. The intuition is described in the following example: denote by *i* a β-agent supporting a link with an α-agent, denoted by *j*; agent *j* does not receive any payoff from the passive link, while switching action she will receive an externality of *e*. That is, passive links from

[18] In the single game is $1-e$

β-agents generate externalities only to β-agents. So in a first approximation, there is an incentive of *e* to switch from action α to action β for each passive link from β-agents. Viceversa a passive link from an α-agent generates externalities (of different size) for all kinds of agents, of *1* for α-agents, of *e* for β-agents. So in a first approximation, there is an incentive of $1-e$ to switch from action β to action α for each passive link from α-agents. Given that $e>0,5$ passive links generate more incentives to switch from action α to action β that viceversa.

A third effect affecting stochastically stable states is the link cost. Indeed, for any given value of *e*, efficient states are stochastically stable only for intermediate values of *k*. The intuition is that, for small values of *k*, the advantage to be coordinated on the efficient states, deriving from a smaller number of links, is lower. On the other side for large values of *k*, coordination problems seem to play an important role to rule out the efficient states.

5. Conclusion

In this paper we have analyzed in a stylized form a social network characterized by an endogenous level of decay that is assumed to depend on the actions chosen by agents participating to the network. Differently from Hojman and Szeidl [13], Jackson and Watts [15], Goyal and Vega Redondo [10], our model is characterized by equilibrium network architectures that depend on which action agents coordinate.

In this model we have a large number of equilibria and we are not able to produce a full description of them; on the contrary we are able to produce an almost full characterization of the set of stochastically stable states. The main result is that efficient states are stochastically stable for intermediate levels of link cost and if the premium to play efficient action is sufficiently high.

Further development can be made in many directions. First, we might consider a model with two-side link formation: this is more similar to real world and it may change the result on stochastic stability. Second, we might use a setting where small deviations from the best response are more likely that large ones. Third, we can model the endogenous decay using different social games that could be better fit to different empirical situations.

Bibliografia

[1] V. Bala, S. Goyal, A Non Cooperative Model of Network Formation, Econometrica 68 (2000), 1181-1231.

[2] V. Bhaskar, F. Vega-Redondo, Migration and the Evolution of Conventions, J. Econ. Behav. Organ. 55 (2004), 397-418.

[3] Y. Bramoullé, D. Lopez-Pintado, S. Goyal, F. Vega-Redondo, Network formation and anti-coordination games, Int. J. Game Theory 33 (2004), 1-19.

[4] Droste, Gilles, Johnson (2000): "Evolution of Conventions in Endogenous Social Networks", Virginia Tech.

[5] G. Ellison, Learning, Local Interaction and Coordination, Econometrica 61 (1993), 1047-1071.

[6] G. Ellison, Basins of Attraction, Long Run Stochastic Stability and the Speed of Step-by-step Evolution, Rev. Econ. Stud. 67 (2000), 17-45.

[7] F. Feri (2006): Stochastic stability in networks with decay, JET Forthcoming

[8] D. Foster, H.P. Young, Cooperation in the Short and in the Long Run, Games Econ. Behav. 3 (1991), 145-156.

[9] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical System, New York, Springer-Verlag (1984).

[10] S. Goyal, F. Vega-Redondo, Network Formation and Social Coordination, Games Econ. Behav. 50 (2005), 178-207.

[11] J.C. Harsanyi, R. Selten, A General Theory of Equilibrium in Games, MIT Press, Cambridge, MA, (1988).

[12] D. Hojman and A. Szeidel, Core and Periphery in Endogenous Networks, KSG Faculty Research Working Paper Series RWP06-022, June 2006.

[13] D. Hojman, A. Szeidl, Endogenous networks, social games, and evolution, Games Econ. Behav. 55 (2006), 112-130.

[14] M.O. Jackson, The Stability and Efficiency of Economic and Social Networks, in: B. Dutta, M.O. Jackson (Eds.), Models of the Formation of Networks and Groups, Springer-Verlag, Heidelberg, (2001).

[15] M.O. Jackson, A. Watts, On the Formation of Interaction Networks in Social Coordination Games, Games Econ. Behav. 41 (2002), 265-291.

[16] M.O. Jackson, A. Watts, The Evolution of Social and Economic Networks, J. Econ. Theory 106 (2002), 265-295.

[17] M.O. Jackson, A. Wolinsky, A Strategic Model of Social and Economic Networks, J. Econ. Theory 71 (1996), 44-74

[18] M. Kandori, R. Rob, Evolution of Equilibria in the Long Run: A General Theory and Applications, J. Econ. Theory 65 (1995), 383-414.

[19] M. Kandori, G.J. Mailath, R. Rob, Learning, Mutation and Long Run Equilibria in Games, Econometrica 61 (1993), 29-56.

[20] R. Karandikar, D. Mookherjee, D. Ray, F. Vega-Redondo, Evolving Aspirations and Cooperation, J. Econ. Theory 80 (1998), 292-331.

[21] A.J. Robson, F. Vega-Redondo, Efficient Equilibrium Selection in Evolutionary Games with Random Matching, J. Econ. Theory 70 (1996), 65-92.

[22] L. Samuelson, Evolutionary Games and Equilibrium Selection, MIT Press, Cambridge, MA, (1997).

[23] R. Selten, Evolution, Learning and Economic Behavior, Games and Economic Behavior 3 (1991), 3-24.

[24] F. Vega-Redondo, Economics and the Theory of Games, Cambridge University Press, (2003).

[25] A. Watts, A Dynamic Model of Network Formation, Games Econ. Behav. 34 (2001), 331-341.

[26] H.P. Young, The Evolution of Conventions, Econometrica 61 (1993), 57-84.

[27] H.P. Young, The Economics of Convention, J. Econ. Perspect. 10 (1996), 105-112.

Appendix.

Proof of Lemma 1.

<u>Part *a*</u> ($k<1$). In this range of link's cost the best response of any α-agent is to be tied (either directly or indirectly) with all other α-agents in unique α-group because $1-k>0$. Then after the first agent has revised, the network will be connected. We note that agents have incentive to delete all links that are not necessary to maintain the network connected. Indeed it is directly verifiable that deleting these links the payoff of supporting agents increases of k for each deleted link. Then after all agents have revised, the network will be minimally connected. Finally we note that agents have no incentive to switch to action β because it causes a payoff reduction of $1-e$ for each connected agent.

<u>Part *b*</u> ($k>1$). From any $s \in S$ such that $a_i = \alpha \ \forall i \in N$ there is a postive probability to go in a state $s \in S$ such that $g = g^e$. The proof of this result is omitted because it uses the same arguments than in Theorem 4.1 in Bala and Goyal [1]. We note that agents characterized by a best response to support at least one link and those receiving at least one passive link have no incentive to switch to action β because it causes a payoff reduction of at least $1-e$ for each connected (either directly or indirectly) agent. Only no-connected agents could switch to action β. Moreover we note there is no incentive to support links with no-connected agents. These evidences are sufficient to prove that there is zero probability that system converges to a ste characyerized by a coonected network and

$a_i = \beta$ for some $i \in N$. QED.

Proof of Lemma 2.

<u>Part a</u> ($k < e - e^2$). The best response of β-agents is to be directly tied with all other agents because $e - k > e^2$; to change action is not a best response because produces a zero payoff. It is directly verifiable that, after all agents have revised, the network will be a $g \in G^c$.

<u>Part b</u> *($e - e^2 < k < e$).* The best response of β-agents is to be either directly or indirectly tied with all other agents because $e - k > 0$; to change action is not a best response because produces a zero payoff. Therefore it is directly verifiable that, after all agents have revised, the network will be connected.

<u>Part c</u> *($e < k < 1$).* Given any $g \in G$ we define the following sets of agents: $L(i;g) = \{i \in N : g_{ij} = 0\ \forall\, j \in N\}$, $M(i;g) = \{i \in N : g_{ij} = 1 \text{ at least for one } j \in N, j \neq i\}$

Give the chance to revise only to agents $i \in M$. After each revision $|M|$ decreases or does not change while $|L|$ increases or does not change. Therefore the dynamic process converges either: *a*) in a state where M is empty ($g = g^e$) or *b*) in a state where $|M| > 0$ and does not change.

Suppose case *a)*. We note that a state characterized by $g = g^e$ cannot be an equilibrium. Indeed, in this state, agents with the chance to revise choose the action randomly and when happen that one chooses action α, all revising agents will have as best response to choose action α and to be tied in an unique α-group. So the state will be characterized by $a_i = \alpha\ \forall i \in N$ and $g \in G^m$.

Suppose case *b)*. In this case $\forall i \in M$ obtains a positive net payoff from her (link) strategy. Now suppose more than one component; each agent can profitably add to its current links (if any) the links supported by any agent $i \in M$ in another component and, by doing so, obtains an additional payoff. Then, after all agents have revised, the network will be connected.

<u>Part d</u> ($1 < k < e + (n-2)\cdot e^2$). The proof follows the same reasoning of part *c*, then it is omitted. We note that the probability to move from any state $s \in S$ such that $g = g^e$ to states $s \in S$ such that $g \neq g^e$ is zero.

<u>Part e</u> ($k > e + (n-2)\cdot e^2$). This proof is based on the observation that the link cost is higher of the maximum payoff obtainable from a single link. QED.

Proof of Lemma 3

Start with a state $s' \in S_\alpha$ and consider an agent $i_1 \in N$ that changes strategy by choosing her corresponding strategy in any $s'' \in S_\alpha$. Then, if i_1 obtains the chance to revise her strategy after the other agents have revised, we obtain another type of $s \in S_\alpha$ where agent i_1 has the same profile of link decisions as in s''. We denote this new state by s^1 and we note that network is minimally connected. Consider an agent $i_2 \in N/\{i_1\}$ who changes strategy by choosing her corresponding strategy in s''. We note that after this mutation the link between i_1 and i_2 is supported as in g''. If i_1 and i_2 obtain the chance to revise their strategy after the other agents have done so, we obtain another type of $s \in S_\alpha$, denoted by s^2, where agents i_1 and i_2 have the same profile of link decisions as in s''. Consider an agent $i_3 \in N/\{i_1, i_2\}$ that changes strategy by choosing her corresponding strategy in s''. After this mutations the links between i_1, i_2 and i_3 are supported as in g''. If i_1, i_2 and i_3 obtain the chance to revise their strategy after the other agents have revised, we obtain another type of $s \in S_\alpha$ where agents i_1, i_2 and i_3 have the same profile of link decisions as in s^2. In this way, we can find a path of one step mutations, which produce the transition between two generic types of $s \in S_\alpha$. QED.

Proof of Lemma 4

Denote by q_i^h, $h \in \{\alpha, \beta\}$, the number of links that agent *i* supports to agents choosing action *h*. Similarly, r_i^h stands for the number of passive links received from agents choosing action *h*.

<u>Step I.</u> Consider the transition from some $s \in S_\alpha$ to some $s' \in S_\beta$. Suppose that in any state $s \in S_\alpha$ x agents switch randomly to some strategy $s_i = (g_i, \beta)$ and denote by $M_{\alpha\beta}$ the set of these agents. For any agent $i \in N/M_{\alpha\beta}$ the best payoff from choosing action β is given by:

(A.1) $$\Pi_i(\beta) = (n-1)\cdot e - \left(L + q_i^\beta\right)\cdot k$$

where L denotes the number of α-groups do not linked to agent *i.* It is directly verifiable that the best response of agent *i* is to support one link for each agent $j \in M_{\alpha\beta}$ such that $g_{ji} = 0$ and one link for each α-group that is no linked with her. The payoff from choosing action α is equal to:

(A.2) $$\Pi_i(\alpha) = (n-1-x) - L\cdot k$$

In this case agent *i* receives a strictly positive payoff only from agents choosing α, then her best response is to support one link for each α-group that is no linked with her and not to support any link with all others. Agent *i* prefers action β if and only if $\Pi_i(\beta) - \Pi_i(\alpha) \geq 0$ that rewritten is:

(A.3) $$x \geq q_i^\beta \cdot k + (n-1)\cdot(1-e)$$

It is directly verifiable that the minimum number of mutations needs to induce agent *i* to switch to action β is when all agents $j \in M_{\alpha\beta}$ are supporting a link with her, that implies $q_i^{\beta} = 0$. In this case minimum number of mutations needs to induce agent *i* to switch to action β is given by:

(A.4) $$m_{\alpha\beta} = \lceil (n-1) \cdot (1-e) \rceil$$

Now we show as $m_{\alpha\beta}$ mutations are also sufficient to induce a transition from some $s \in S_{\alpha}$ to some $s' \in S_{\beta}$ through the ensuing operations of unperturbed dynamics alone. Assume that all $i \in M_{\alpha\beta}$ support links with all $j \in N / M_{\alpha\beta}$. In this case the best response of any $j \in N / M_{\alpha\beta}$ is to switch to action β. Then giving the chance to revise to agents $i \in M_{\alpha\beta}$ after all $j \in N / M_{\alpha\beta}$ have revised, the system transits to some state in S_{β}. From (A.4) it is directly verifiable that $m_{\alpha\beta} \geq 2$ for values of *n* large enough.

Step II. Consider the transition from some $s \in S_{\beta}$ to some $s' \in S_{\alpha}$. Suppose that in a state $s \in S_{\beta}$ x agents switch to some strategy $s_i = (g_i, \alpha)$ and denote by $M_{\beta\alpha}$ the set of these agents. For any agent $i \in N / M_{\beta\alpha}$ the best payoff from choosing action β is given by (A.1) while that from choosing action α is:

(A.5) $$\Pi_i(\alpha) = x - L \cdot k$$

Agent *i* prefers action α if and only if $\Pi_i(\alpha) - \Pi_i(\beta) \geq 0$ that rewritten is:

(A.6) $$x \geq (n-1) \cdot e - q_i^{\beta} \cdot k \, .$$

It is directly verifiable that the minimum number of mutations needs to induce agent *i* to switch to action α is for an agent *i* supporting links with all $j \in N / M_{\beta\alpha}$, that implies $q_i^{\beta} = n - 1 - x$. In this case the minimum number of mutations needs to induce agent *i* to switch to action α is given by:

(A.7) $$m_{\beta\alpha} = \left\lceil (n-1) \cdot \frac{(e-k)}{1-k} \right\rceil .$$

Now we show as $m_{\beta\alpha}$ mutations are also sufficient to induce a transition from some $s \in S_{\beta}$ to some $s' \in S_{\alpha}$ through the ensuing operations of unperturbed dynamics alone. Suppose a state $s \in S_{\beta}$ in which every agent $i = 1, 2, ..., n$ supports links with all $j > i$ and assume that agents $1, 2, ..., m_{\beta\alpha}$ have switched to action α. If the chance to revise is given to all agents $i > m_{\beta\alpha}$ in order (according to the index *i*), condition minimizing (A.6) is satisfied for all revising agents, indeed $x = i - 1 \geq m_{\beta\alpha}$ and $q_i^{\beta} = n - 1 - x$. Therefore it is directly verifiable that condition (A.6) is satisfied for all agents $i \geq m_{\beta\alpha} + 1$. After all agents $i > m_{\beta\alpha}$ have revised, the convergence to some state $s \in S_{\alpha}$ is proved using the result in Lemma 2. From (A.7) it is directly verifiable that $m_{\beta\alpha} \geq 2$ for values of n large enough.

Step III. Consider the difference $m_{\alpha\beta} - m_{\beta\alpha}$. Using expressions (A.4) and (A.7) we find that $m_{\alpha\beta} > m_{\beta\alpha}$ if $\left[(n-1)\frac{1-2e+ek}{1-k}\right] > 0$; this condition verified when $k > 2 - \frac{1}{e}$ and n is sufficiently large; otherwise $m_{\alpha\beta} < m_{\beta\alpha}$ if $k < 2 - \frac{1}{e}$ and n is sufficiently large. QED.

Proof of Lemma 5

Consider any $s \in S$ such that $a_i = \beta \forall i \in N$ and $g \in G^s$. Suppose a mutation in which an agent changes link strategy but does not switch action. We note that in this case no agent has incentive to switch action; indeed, using action β any agent obtains a strictly positive payoff from any link strategy; on the contrary, switching to action α any agent obtains a zero payoff. Therefore for this kind of mutation the proof is equal to that of Lemma 3 in Feri [7] and it is omitted. Now consider a mutation where the "mutant" agent changes both link strategy and action. We have to prove the convergence in the following cases: *a)* mutation of the central agent, denoted by *c*; *b)* mutation of a peripheral agent, denoted by *m*. Consider case (*a*), where agent *c* switches to action α and chooses randomly any link strategy. We note that for any link strategy chosen by agent *c*, all other agents do not receive any indirect payoff. Therefore, if *n* is sufficiently large, the best response of the first agent with the chance to revise is to have a direct link with all other agents[19]. When other agents have the chance to revise their strategy, they delete all supported links because they are optimally (indirectly) linked with all others. When agent *c* obtains the chance to revise, he switches to action β and deletes all supported links. Therefore, after all agents have revised their strategy the state will be characterized by $a_i = \beta \ \forall i \in N$ and $g \in G^s$. Consider case (*b*), where a peripheral agent (denoted by *m*) switches to action α and chooses randomly any link strategy. After this mutation, the states can be summarized in the following two sub-cases: *b1)* agent *m* is no connected; *b2)* agent *m* is connected. Now consider case (*b1*). All agents $i \in N / m$ are connected in a star network (with agent *c* at the center). If the chance to revise the strategy arrives to agent *c*, her best response is to support a new link with agent *m* (she obtains an additional payoff of $e-k$). In this new state the

[19] Suppose the revising agent is receiving a link from *c*; if she switches to action α, she obtains a payoff of 1; if she continues to use action β and supports link with all others, she obtains a payoff of $e + (e-k)\cdot(n-2)$; this last case is a best response if $n > 1 + (1-k)/(e-k)$. It is directly verifiable that, when the revising agent is supporting the link with *c*, the condition is the same.

network will be a star and it is directly verifiable that the best response of $\forall i \in N/m$ is not to change strategy. Therefore agent *m* switches to action β as soon as possible. If the chance to revise the strategy arrives to a peripheral agent, her best response is to support a new link with agent *m* (she gains an additional payoff of $e-k$). In this new state the best response of $\forall i \in N/m$ is not to change strategy. Then, when the chance to revise the strategy arrives to *m*, she switches to action β and, if *n* is sufficiently large, supports a new link with agent *c*.[20] Finally in this new state the only agent with an incentive to change strategy is the agent supporting a link with *m* (the first revising agent): as soon as possible she deletes that link (increasing her payoff of $e^2-(e-k)$). In both cases discussed above the final state will be characterized by $a_i = \beta\ \forall i \in N$ and $g \in G^s$. Now consider case (*b2*) in which, after the mutation, agent *m* remains connected in the network. Note that all agents $i \in N/m,c$ are directly linked with *c* and agents do not receive indirect payoff from a link with *m*. Moreover $d(i,j)=2\ \forall ij \in N/m,c$ and either $d(i,m)=2\ \forall i \in N/c$ or $d(i,m)=3\ \forall i \in N/c$ depending on agent *m* is directly linked with *c* or not. From these considerations it follows that all agents $i \in N/m$ have no incentive to change strategy. Therefore, when agent *m* receives the chance to revise she switches to her ante-mutation strategy and the state goes back to the initial one. QED.

Proof of Lemma 6

Let q_i^h and r_i^h, $h \in \{\alpha,\ \beta\}$, defined as in Lemma 4.

<u>Step I.</u> Consider the transition from any $s \in S_\alpha$ to any $s' \in S_\beta$. Suppose that in any $s \in S_\alpha$ x agents switch randomly to some strategy $s_i = (g_i, \beta)$ and denote by $M_{\alpha\beta}$ the set of these agents. For any agent $i \in N/M_{\alpha\beta}$ the best payoff from choosing action β is given by:

$$\text{(A.8)} \qquad \Pi_i(\beta) = \left(r_i^\beta + q_i^\beta + y_{i,1}^\alpha\right)\cdot e + \sum_{d=2}^{\bar{d}}\left(y_{i,d}^\alpha + y_{i,d}^\beta\right)\cdot e^d - \left(q_i^\alpha + q_i^\beta\right)\cdot k$$

where $y_{i,d}^\alpha$ is the number of α-agents belonging to α-groups indirectly linked with agent *i* through *d-1* β-agents [21], $y_{i,d}^\beta$ is the number of β-agents indirectly linked with agent *i* through *d-1* β-agents and $\bar{d}$ is the maximum integer such that if $d \le \bar{d}$ then $e-k < e^d$. Note that

[20] Choosing action β and without a link with *c*, agent *m* receives a payoff of $e+e^2+(n-3)\cdot e^3$; supporting a new link with *c* , agent *m* receives $2\cdot e+(n-3)\cdot e^2-k$. Then if $n > 3+\left[k-\left(e-e^2\right)\right]\big/\left(e^2-e^3\right)$ to support a direct link with *c* is a best response.

[21] Note, by $y_{i,d}^\alpha$ we denote the number of α-agents belonging to α-groups that are directly linked with agent *i*

$q_i^\beta + r_i^\beta + y_{i,1}^\alpha + \sum_{d=2}^{\bar{d}} \left(y_{i,d}^\alpha + y_{i,d}^\beta \right) = n-1$ and $r_i^\beta + q_i^\beta + \sum_{d=2}^{\bar{d}} y_{i,d}^\beta = x$. By $Ly_{i,d}^\alpha$ denote the number of α-groups to which the $y_{i,d}^\alpha$ agents belong. When agent *i* is optimally linked, $y_{i,d}^\alpha > 0$ implies that:

(A.9) $$y_{i,d}^\alpha \cdot e^d > y_{i,d}^\alpha \cdot e - k \cdot Ly_{i,d}^\alpha,$$

otherwise agent *i* could increase her payoff supporting a direct link for each of the $Ly_{i,d}^\alpha$ α-groups. On the other hand, the payoff from choosing α is equal to:

(A.10) $$\Pi_i(\alpha) = n - 1 - x - \left(q_i^\alpha + \sum_{d=2}^{\bar{d}} Ly_{i,d}^\alpha \right) \cdot k$$

The agent *i* prefers action β if and only if $\Pi_i(\beta) - \Pi_i(\alpha) \geq 0$, that rewritten is:

(A.11) $$x \geq (n-1) - y_{i,1}^\alpha \cdot e - \sum_{d=2}^{\bar{d}} y_{i,d}^\alpha \cdot e^d - k \cdot \sum_{d=2}^{\bar{d}} Ly_{i,d}^\alpha - r_i^\beta \cdot e - q_i^\beta \cdot (e-k) - \sum_{d=2}^{\bar{d}} \left(y_{i,d}^\beta \right) \cdot e^d$$

The minimum number of mutations needs to induce agent *i* to switch to action β is given by the network minimizing the right part of (A.11). It happens when:

1) $y_{i,2}^\alpha = n - 1 - x$, $y_{i,d}^\alpha = 0 \ \forall d \neq 2$ and $Ly_{i,d}^\alpha = y_{i,d}^\alpha$, that is, all α-agents are indirectly linked to agent *i* through one β-agent and between α-agents there are no links.

2) $r_i^\beta = x$, $q_i^\beta = 0$ and $y_{i,d}^\beta = 0 \forall d$, that is all β-agents support a link with *i*.

Then, using these conditions and solving (A.11) in x ,we find the minimum number of mutations needs to induce agent *i* to switch to action β, that is given by:

(A.12) $$m_{\alpha\beta} = \left\lceil (n-1) \cdot \frac{1 - e^2 - k}{1 + e - e^2 - k} \right\rceil$$

Now we show as $m_{\alpha\beta}$ mutations are also sufficient to induce a transition from some $s \in S_\alpha$ to some $s' \in S_\beta$ through the ensuing operations of unperturbed dynamics alone. Start with an $s \in S_\alpha$ such that $g \in G^{cs}$ and suppose: *a*) $m_{\alpha\beta}$ agents switch to action β, *b*) the central agent of the star (denoted by *c*) belongs to $M_{\alpha\beta}$, *c*) all $i \in M_{\alpha\beta}$ support links with all $j \in N / M_{\alpha\beta}$. In this case the best response of any $j \in N / M_{\alpha\beta}$ is to switch to action β because are verified the conditions minimizing the right part of (A.11). Note that when agents $j \in N / M_{\alpha\beta}$ revise, their best link's strategy is not to support any link because they are optimally linked (directly or indirectly through agent *c*) to all others. Then giving the chance to revise to agents $i \in M_{\alpha\beta} / c$ after all $j \in N / M_{\alpha\beta}$ have revised (and switched to action β), it is directly verifiable that they sever all supported links because are optimally linked through agent *c*. Therefore the resulting state will be $s \in S_\beta$ such that

$g \in G^{cs}$. From (A.12) it is directly verifiable that $m_{\alpha\beta} \geq 2$ if $k < 1 - e^2$ and n is large enough, otherwise, if $k \geq 1 - e^2$, it is not possible to find a sufficiently large value of n such that $m_{\alpha\beta} > 1$.

Step II. Consider the transition from some $s \in S_\beta$ to some $s' \in S_\alpha$. Suppose that in any state $s \in S_\beta$ x agents switch randomly to some strategy $s_i = (g_i, \alpha)$ and denote by $M_{\beta\alpha}$ the set of these agents. For any agent $i \in N / M_{\beta\alpha}$ the best payoff from choosing action β is given by (A.8) with the difference that now is $\sum_{d=1}^{\bar{d}} y_{i,d}^{\alpha} = x$. The best payoff from choosing action α is:

(A.13) $$\Pi_i(\alpha) = x - \left(q_i^{\alpha} + \sum_{d=2}^{\bar{d}} L y_{i,d}^{\alpha} \right) \cdot k$$

Agent i prefers action α if and only if $\Pi_i(\alpha) - \Pi_i(\beta) \geq 0$, that rewritten is:

(A.14) $$x > k \cdot \sum_{d=2}^{\bar{d}} L y_{i,d}^{\alpha} + \sum_{d=2}^{\bar{d}} y_{i,d}^{\alpha} \cdot e^d + y_{i,1}^{\alpha} \cdot e + r_i^{\beta} \cdot e + \sum_{d=2}^{\bar{d}} y_{i,d}^{\beta} \cdot e^d + q_i^{\beta}(e - k)$$

The minimum number of mutations needs to induce agent i to switch to action α is given by the network minimizing the right part of (A.14). It happens when:

1) $y_{i,1}^{\alpha} = x$, $y_{i,d}^{\alpha} = 0 \; \forall d > 1$, that is all α-agents are directly linked to i.

2) $r_i^{\beta} = 0$, $q_i^{\beta} = n - 1 - x$ and $y_{i,d}^{\beta} = 0 \forall d$, that is agent i supports links with all β-agents.

Then using these conditions and solving (A.14) in x we find the minimum number of mutations needs to induce agent i to switch to action α that is given by:

(A.15) $$m_{\beta\alpha} = \left\lceil (n-1) \cdot \frac{e-k}{1-k} \right\rceil$$

Now we show as $m_{\beta\alpha}$ mutations are also sufficient to induce a transition from some $s \in S_\beta$ to some $s' \in S_\alpha$ through the ensuing operations of unperturbed dynamics alone. Start with an $s \in S_\beta$ such that $g \in G^c$ and suppose $m_{\beta\alpha}$ agents switch to action α and that the central agent of the star (denoted by c) belongs to $i \in M_{\beta\alpha}$. In this case the best response of any $j \in N / M_{\beta\alpha}$ is to switch to action α because are verified the conditions minimizing the right part of (A.14). Indeed in this case, the best response of any agent $j \in N / M_{\beta\alpha}$, conditional to $a_j = \beta$, is to support a new link for each β-agent (then one link for each $z \in N / (M_{\alpha\beta} \cup j)$ and either to support or to maintain a direct link with agent c. It implies $y_{i,1}^{\alpha} = x$ and $q_i^{\beta} = n - 1 - x$. The best response of any agent $j \in N / M_{\beta\alpha}$, conditional to $a_j = \alpha$, is either to support or to maintain a direct link with agent c. Therefore it is directly verifiable that giving the chance to revise to all agent $j \in N / M_{\beta\alpha}$ the system converges to a state $s \in S_\alpha$ such that $g \in G^c$. From (A.15) it is directly verifiable that $m_{\alpha\beta} \geq 2$ if n is large enough.

<u>Step III.</u> Finally we consider the difference $m_{\alpha\beta} - m_{\beta\alpha}$. Using expressions (A.12) and (A.15) we find that $m_{\alpha\beta} - m_{\beta\alpha} > 0$ if $k > \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$ and n is sufficiently large[22]; otherwise $m_{\alpha\beta} - m_{\beta\alpha} < 0$ if $k < \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$ and n is sufficiently large. QED.

Proof of Lemma 7

Consider any state $s \in S$ such that $a_i = \beta \; \forall i \in N$ and suppose a mutation in which an agent m forms links with all others and does not change action. When agents $i \in N / m$ have the chance to revise, they delete all supported links (note that they do not change action because any agent, switching action, obtains a payoff equal to zero, while choosing β obtains a strictly positive payoff). After all agents $i \in N / m$ have revised, the state will be characterized by a $g \in G^{cs}$ with agent m at the center. In this state agent m obtains a strictly negative payoff equal to $(n-1) \cdot (e-k)$. When agent m has the chance to revise, she will sever all supported links and the state will be characterized by $g = g^e$ and $a_i = \beta \; \forall i \in N$. We note that in this state any revising agent is indifferent to chose an action or another because she obtains, in any case, a payoff equal to zero. Then, when an agent receives the chance to revise, he will choose the action randomly. After a first agent has switched to action α, the best response of all other agents will be to switch to action α and to support one (new) link with any α-agents. Then giving the chance to revise to all agents the state will go in any $s \in S_\alpha$. QED.

Proof of Lemma 8

This results is strictly related to Lemma 6. When $m_{\alpha\beta} > 1$ (one mutation is not enough to cause a transition from any $s \in S_\alpha$ to $s \notin S_\alpha$), using equation (A.12), we obtain $k < 1 - e^2 - \frac{e}{n-2}$. QED

Proof of Lemma 9

Consider any state $s \in S$ such that $a_i = \beta \; \forall i \in N$ and suppose a mutation in which an agent m forms links with all others and does not change action. When agents $i \in N / m$ have the chance to

[22] The necessary condition for $m_{\alpha\beta} > 1$ is $k < 1 - e^2$; it is always satisfied for all $k > \frac{e + 2 \cdot e^2 - e^3 - 1}{2 \cdot e - 1}$ such that $e - e^2 < k < e$.

revise, they delete all supported links (note that they do not change action because any agent, switching action, obtains a payoff equal to zero, while choosing β obtains a strictly positive payoff). After all agents $i \in N/m$ have revised, the state will be characterized by a $g \in G^{cs}$ with agent *m* at the center. In this state agent *m* obtains a strictly negative payoff equal to $(n-1)\cdot(e-k)$. When agent *m* has the chance to revise, she will sever all supported links and the state will be characterized by $g = g^e$ and $a_i = \beta \ \forall i \in N$. In this state the best response of any agent is not to support any link, because she could receive only a strictly negative payoff. QED.

Proof of Lemma 10

Consider any state $s \in S$ such that $g = g^e$. The best-response of an agent supporting links with agents that are not linked with any other, is to sever such links. Consider any state characterized by g^e and suppose a mutation in which an agent *m* forms *x* links ($0 < x < n-1$) and chooses action β. The best-response of non connected agents is to choose action β and to form a link with *m* if:

(A.16) $$x \geq (k-e)/e^2$$

otherwise their best-response is not to form links and, when agent *m* has a new chance to revise, the system goes in a state characterized by g^e. If (A.16) is true and agent *m* has the chance to revise her strategy after *y* non connected agents have done ($0 < y < n-x$), the state will be characterized by a g^{ps} with *y+1* β-agents (agent *m* severs all supported links with peripheral agents). Follows that best-response of non connected agents is to support a link with *m* and to choose action β if:

(A.17) $$y \geq (k-e)/e^2$$

Then, if (A.17) is true, after all non connected agents have revised their strategy, the resulting state will be characterized by a (connected) g^{ps} and $a_i = \beta \ \forall i \in N$, otherwise (if (A.17) is false) the state will be characterized by a g^e; indeed in this second case the best-response of all agents is not to support any links. Then, to transit in a state where $a_i = \beta \ \forall i \in N$, both conditions (A.16) and (A.17) have to be true (agents receiving the link from *m* have to be different from agents supporting link with him). A necessary condition is a sufficiently large number of agents, that is $n \geq 1 + x + y$. Then using conditions (A.16) and (A.17) and solving by *k* the necessary condition is:

(1.18) $$k < e + \frac{n-1}{2} \cdot e^2 \ .$$

Therefore if (1.18) is not satisfied one mutation is not enough to transit in an absorbing state different from the initial one. QED.

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Network formation with endogenous decay

Abstract

This paper considers a model of economic network characterized by an endogenous architecture and frictions in the relations among agents as described in Bala and Goyal (2000). We propose a similar network model with the difference that frictions in the relations among agents are endogenous. Frictions are modeled as dependent on the result of a coordination game, played by every pair of directly linked agents and characterized by 2 equilibria: one efficient and the other risk dominant. The model has a multiplicity of equilibria and we produce a characterization of those are stochastically stable.

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